13.5. Wednesday for MAT3006

13.5.1. Fubini's and Tonell's Theorem

Motivation. Given two measurable space $(\mathbb{R}, \mathcal{M}, \mathrm{d}x)$ and $(\mathbb{R}, \mathcal{M}, \mathrm{d}y)$, we have constructed the product measurable space $(\mathbb{R}^2, \mathcal{M} \otimes \mathcal{M}, \mathrm{d}\pi)$. Suppose $f : \mathbb{R}^2 \to [-\infty, \infty]$ is measurable on this space, now we want to show that

$$\int f(x,y)\,\mathrm{d}\pi = \int \left(\int f_y(x)\,\mathrm{d}x \right) \mathrm{d}y = \int \left(\int f_x(y)\,\mathrm{d}y \right) \mathrm{d}x$$

Easier Goal. The proof for the statement above is hard. Consider the easier case where f is a simple function first, i.e., $f(x,y) = \mathcal{X}_E(x,y), E \in \mathcal{M} \otimes \mathcal{M}$, which follows that

$$\begin{aligned}
&\int \mathcal{X}_E(x,y)\,\mathrm{d}\pi = \pi(E) \\
&\int (\mathcal{X}_E)_y(x)\,\mathrm{d}x = \int \mathcal{X}_{E_y}(x)\,\mathrm{d}x = m_X(E_y) \\
&\int (\mathcal{X}_E)_x(y)\,\mathrm{d}y = \int \mathcal{X}_{E_x}(y)\,\mathrm{d}x = m_Y(E_x)
\end{aligned}$$

Therefore, our easier goal is to show that

$$\pi(E) = \int m_X(E_y)\,\mathrm{d}y = \int m_Y(E_x)\,\mathrm{d}x, \quad \forall E \in \mathcal{M} \otimes \mathcal{M}. \tag{13.1}$$

Easiest Goal. Consider the simplest case where $E = A \times B \in \mathcal{M} \otimes \mathcal{M}$, where $A \in \mathcal{M}_X, B \in \mathcal{M}_Y$, which implies

- $\pi(A \times B) = m_X(A) m_Y(B)$

- As shown in the figure (13.1), for fixed $y \in Y$,

$$m_X((A \times B)_y) = \begin{cases} m_X(A), & \text{if } y \in B \\ m_X(\emptyset) = 0, & \text{if } y \notin B \end{cases} = m_X(A)\mathcal{X}_B(y)$$

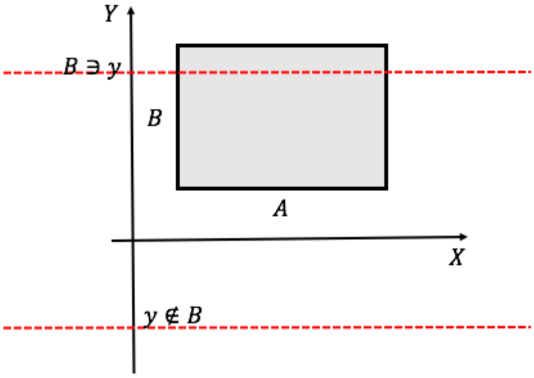

Figure 13.1: Illustration for $m_X((A \times B)_y)$

Therefore, we imply

$$\begin{aligned}\int m_X((A \times B)_y)\,\mathrm{d}y &= \int m_X(A)\mathcal{X}_B(y)\,\mathrm{d}y \\ &= m_X(A)\int \mathcal{X}_B(y)\,\mathrm{d}y \\ &= m_X(A)m_Y(B)\end{aligned}$$

Similarly,

$$\int m_Y((A \times B)_x)\,\mathrm{d}x = m_X(A)m_Y(B).$$

Therefore, the easier goal (Eq. (13.1)) holds for $E = A \times B \in \mathcal{M} \times \mathcal{M}$.

R Generalization from the easier goal to the real goal is trivial, i.e., applying MCT is ok. The difficulty is that how to show the easier goal (Eq. (13.1)) holds for any $E \in \mathcal{M} \otimes \mathcal{M}$, given that the easier goal (Eq. (13.1)) holds for any $E \in \mathcal{M} \times \mathcal{M}$.

Definition 13.2 [Monotone Class] Let X be a non-empty set. A **monotone class** $\mathcal{T}$ is a collection of subsets of X closed under countable increasing unions and countable decreasing intersections, i.e.,

1. If $E_i \in \mathcal{T}(i \in \mathbb{N})$ and $E_i \subseteq E_{i+1}, \forall i$, then

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{T}$$

2. If $F_i \in \mathcal{T}(i \in \mathbb{N})$ with $F_i \supseteq F_{i+1}, \forall i$, then

$$\bigcap_{i=1}^{\infty} F_i \in \mathcal{T}$$

■

R Every σ-algebra is a monotone classs. In particular, for $X = \mathbb{R}$, the collection of subsets $\mathcal{M}$ and $\mathcal{B}$ are both monotone classes.

Definition 13.3 [Smallest Monotone Class] For any $S \subseteq \mathcal{P}(X)$, denote

$$\mathcal{M}(S) := \bigcap_{\mathcal{T} \text{ is a monotone class such that } S \subseteq \mathcal{T}} \mathcal{T},$$

which is also the monotone class. We call $\mathcal{M}(S)$ as the smallest monotone class containing S. ■

R It's clear that $\mathcal{M}(S) \subseteq \sigma(S)$, where $\sigma(S)$ is the smallest σ-algebra containing S. Question: when do we have $\mathcal{M}(S) = \sigma(S)$?

Theorem 13.5 — Monotone Class Theorem. Let X be a non-empty set. If $S \subseteq \mathcal{P}(X)$ is an **algebra** (i.e., $E_1, E_2 \in S \implies E_1 \cup E_2 \in S, E_1 \cap E_2 \in S, E_1^c \in S$), then $\mathcal{M}(S) = \sigma(S)$.

We skip the proof for the monotone class theorem, but you may refer to the proof in the blackboard.

■ **Example 13.3** 1. Let $X = \mathbb{R}$, and $S^1 = \{\text{all intervals}\}$ is **not** an algebra, e.g.,

$$[1,2] \in S^1 \implies [1,2]^c = (-\infty, 1) \cup (2, \infty) \notin S^1.$$

However, $S = \{\text{finite disjoint union of intervals}\}$ is an algebra. Therefore,

$$\mathcal{M}(S) = \sigma(S) := \mathcal{B}(\text{Borel } \sigma\text{-algebra}).$$

2. Let $X = \mathbb{R}^2$, and define

$$S = \left\{ \text{finite disjoint union of measurable rectangles } \bigcup_{i=1}^{k} (A_i \times B_i) \,\middle|\, A_i, B_i \in \mathcal{M} \right\}$$

Then S is an algebra, for instance, as shown in the Fig. (13.2), $(A \times B)^c = (A^c \times \mathbb{R}) \cup (A \times B^c)$ is a disjoint union of 2 measurable rectangles.

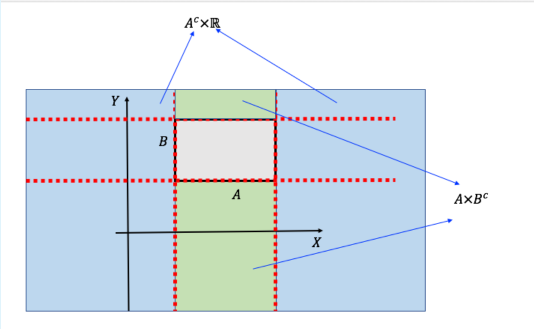

Figure 13.2: Illustration for $(A \times B)^c$

Therefore, $\mathcal{M}(S) = \sigma(S) := \mathcal{M} \otimes \mathcal{M}$

Proposition 13.8 For all $E \in \mathcal{M} \otimes \mathcal{M}$, we have

$$\pi(E) = \int m_Y(E_x)\,\mathrm{d}x = \int m_X(E_y)\,\mathrm{d}y \tag{13.2}$$

Proof. Construct

$$\mathcal{A} = \left\{ E \in \mathcal{M} \otimes \mathcal{M} \,\middle|\, \begin{array}{l} x \mapsto m_Y(E_x) \text{ is a measurable function of } x \\ y \mapsto m_X(E_y) \text{ is a measurable function of } y \\ \text{Eq. (13.2) holds} \end{array} \right\}$$

- Claim 1: $\mathcal{A}$ is a monotone class

- Claim 2: Any finite disjoint union of measurable rectangles is in $\mathcal{A}$:

$$\bigcup_{i=1}^{k}(A_i \times B_i) \in \mathcal{A}, \quad k \in \mathbb{N}$$

If claim (1),(2) holds, then $S \subseteq \mathcal{A}$, where

$$S = \{\text{finite disjoint union of measurable rectangles}\}$$

which follows that

$$\mathcal{M}(S) \subseteq \mathcal{A}.$$

By monotone class theorem, $\sigma(S) = \mathcal{M}(S) \subseteq \mathcal{A}$, i.e.,

$$\mathcal{M} \otimes \mathcal{M} = \sigma(S) = \mathcal{M}(S) \subseteq \mathcal{A} \subseteq \mathcal{M} \otimes \mathcal{M} \implies \mathcal{M} \otimes \mathcal{M} = \mathcal{A}.$$

Therefore, (13.2) holds for all $E \in \mathcal{A} = \mathcal{M} \times \mathcal{M}$.

We left the proof for claim (1) in next class. Now we give a proof for claim (2):

- For any $E = \cup_{i=1}^{k}(A_i \times B_i)$,

$$m_Y(E_x) = \sum_{i=1}^{k} m_Y(B_i)\mathcal{X}_{A_i}(x)$$

  is a simple function on x, and therefore measurable.

- Similarly,

$$m_X(E_y) = \sum_{i=1}^{k} m_X(A_i)\mathcal{X}_{B_i}(y)$$

  is also measurable.

- By the easiest goal, (13.2) also holds.

Therefore, claim (2) is true. ■