13.2. Monday for MAT3006

Notations. In this lecture, we let $\int_I f(x,y)\,\mathrm{d}x$ denote the Lebesgue integral.

Theorem 13.3 Let I, J be intervals in $\mathbb{R}$, and $f : I \times J \to \mathbb{R}$ be a function such that

1. For fixed $y \in J$, the function $f(x) := f(x,y)$ is integrable on I
2. $\frac{\partial f}{\partial y}$ exists for any $(x,y) \in I \times J$
3. $\left|\frac{\partial f}{\partial y}(x,y)\right| \le g(x)$ for some integrable function $g(x)$ on I.

Then $F(y) := \int_I f(x,y)\,\mathrm{d}x$ is differentiable on J, with

$$F'(y) = \int_I \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$$

Proof. Fix $y \in J$, and consider any sequence $\{y_n\}$ (with $y_n \neq y$) in J converging to y.

Construct the function

$$g_n(x) := \frac{f(x,y_n) - f(x,y)}{y_n - y}$$

which follows that

1. The function g_n is integrable by hypothesis (1)
2. The function $g_n(x)$ converges to $\frac{\partial f}{\partial y}(x,y)$ as $n \to \infty$
3. By MVT, $|g_n(x)| = |\frac{\partial f}{\partial y}(x,\xi)|$, which is bounded by $g(x)$ by hypothesis (3).

Therefore, the DCT applies, and

$$\int_I g_n(x)\,\mathrm{d}x = \frac{1}{y_n - y}\left[\int f(x,y_n)\,\mathrm{d}x - \int f(x,y)\,\mathrm{d}x\right] \to \int_I \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$$

In other words, for all sequences $\{y_n\} \to y$ with $y_n \neq y$,

$$\lim_{n\to\infty} \frac{F(y_n) - F(y)}{y_n - y} = \int_I \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$$

From the elementary analysis knowledge, in particular, $\lim_{y'\to y} H(y')$ exists (equal to L) if and only if $\lim_{n\to\infty} H(y_n) = L$ for all sequences $\{y_n\} \to y$ with $y_n \neq y$. Therefore,

$$F'(y) := \lim_{y'\to y} \frac{F(y') - F(y)}{y' - y} = \int_I \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x.$$

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13.2.1. Double Integral

Definition 13.1 [Measure in $\mathbb{R}^2$] In $\mathbb{R}^2$, we can define the **measure** of the rectangle $A \times B \subseteq \mathbb{R}^2$ with $A, B \in \mathcal{M}$ by

$$m^*(A \times B) = m(A)m(B)$$

In particular, we define

$$x \cdot \infty = \infty \cdot x = (-x) \cdot (-\infty) = \begin{cases} \infty, & \text{if } x > 0 \\ -\infty, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

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Definition 13.2 [Outer Measure in $\mathbb{R}^2$] Then the outer measure of any $E \subseteq \mathbb{R}^2$ is defined as

$$m^*(E) := \inf\left\{ \sum_{i=1}^{\infty} m(R_i) \middle| E \subseteq \bigcup_{i=1}^{\infty} R_i, R_i = A_i \times B_i, A_i, B_i \in \mathcal{M} \right\}$$

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Definition 13.3 [Lebesgue Measurable in $\mathbb{R}^2$] A subset $E \subseteq \mathbb{R}^2$ is Lebesgue measurable if E satisfies the Carathedory Property:

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E),$$

for any subset $A \subseteq \mathbb{R}^2$. ■

Construction of Measurable Space in $\mathbb{R}^2$. Given two measurable spaces $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\lambda)$, in particular, we are interested in

$$(X,\mathcal{A},\mu) = (Y,\mathcal{B},\lambda) = (\mathbb{R},\mathcal{M},m).$$

Now we want to construct the measurable space in $X \times Y := \mathbb{R}^2$.

1. Start from the "measurable rectangles"

$$\mathcal{A} \times \mathcal{B} = \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$$

2. Define the function $\pi : \mathcal{A} \times \mathcal{B} \to [0,\infty]$ by

$$\pi(A \times B) = \mu(A)\lambda(B).$$

3. Let $\mathcal{A} \otimes \mathcal{B}$ be the smallest σ-algebra containing $\mathcal{A} \times \mathcal{B}$. Then by Caratheodory extension theorem, we can extend $\pi : \mathcal{A} \times \mathcal{B} \to [0,\infty]$ to $\tilde{\pi} : \mathcal{A} \otimes \mathcal{B} \to [0,\infty]$ such that

   (a) $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \tilde{\pi})$ is a measurable space

   (b) $\tilde{\pi} \mid_{\mathcal{A}\times\mathcal{B}} = \pi$.

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- If further we have $\mathcal{A}$ and $\mathcal{B}$ are σ-finite, i.e., there exists $E_i \in \mathcal{A}$ such that $X = \cup_{i=1}^{\infty} E_i$, $\mu(E_i) < \infty, \forall i$, then we can imply the extension $\tilde{\pi}$ is unique.

  (For instance, $\mathbb{R} = \cup_{n\in\mathbb{Z}}[n,n+1]$ and $m([n,n+1]) = 1 < \infty$, i.e., $(\mathbb{R},\mu,m)$ is σ-finite.)

- Question: we can construct two measurable space $(\mathbb{R}\times\mathbb{R}, \mathcal{M}\otimes\mathcal{M}, \tilde{\pi})$ and $(\mathbb{R}^2, \mathcal{M}_{\mathbb{R}^2}, m)$. Are they the same?

  Answer : no, but the latter can be obtained from the former by completion process. In particular,

$$m \mid_{\mathcal{M}\otimes\mathcal{M}} = \tilde{\pi}.$$

Let's study the measurable space $(\mathbb{R}\times\mathbb{R}, \mathcal{M}\otimes\mathcal{M}, f)$ first, where $f:\mathbb{R}^2\to[-\infty,\infty]$ is a measurable function, i.e., $f^{-1}((a,\infty])\in\mathcal{A}\otimes\mathcal{B}$. In particular, we say $E\subseteq\mathbb{R}\times\mathbb{R}$ is measurable if $E\in\mathcal{M}\otimes\mathcal{M}$ for the moment being (but we will generalize the notion of measurable into $\mathcal{M}_{\mathbb{R}^2}$ in the future).

Definition 13.4 [x-section and y-section] Let $E\subseteq X\times Y$, with $(x,y)\in E$. Define

- the x-section $E_x=\{y\in Y \mid (x,y)\in E\}$, for fixed $x\in X$
- the y-section $E_y=\{x\in X \mid (x,y)\in E\}$, for fixed $y\in Y$.

Proposition 13.2 Suppose that $E\subseteq X\times Y$ is measurable (i.e., $E\in\mathcal{A}\otimes\mathcal{B}$), then $E_x\in\mathcal{B}$ and $E_y\in\mathcal{A}$.

Proof. Construct the set $\mathfrak{A}=\{E\in\mathcal{A}\otimes\mathcal{B} \mid E_x\in\mathcal{B}\}$. It suffices to show $\mathfrak{A}=\mathcal{A}\otimes\mathcal{B}$. We claim that

1. $\mathfrak{A}$ is a σ-algebra
2. $\mathfrak{A}$ contains all $A\times B\in\mathcal{A}\times\mathcal{B}$

If the claim (1) and (2) hold, and since $\mathcal{A}\otimes\mathcal{B}$ is the smallest-σ-algebra containing $\mathcal{A}\times\mathcal{B}$, we imply $\mathcal{A}\otimes\mathcal{B}\subseteq\mathfrak{A}\subseteq\mathcal{A}\otimes\mathcal{B}$, i.e., the proof is complete.

1. (a) Note that $\emptyset\in\mathfrak{A}$, and $X\times Y\in\mathfrak{A}$ since $(X\times Y)_x=Y\in\mathcal{B}$.

   (b) Suppose that $E_i\in\mathfrak{A}$, $i\geq 1$, i.e., $(E_i)_x\in\mathcal{B}$. Observe that

   $$\left(\bigcup_{i=1}^{\infty}E_i\right)_x=\bigcup_{i=1}^{\infty}(E_i)_x\in\mathcal{B},$$

   since $\mathcal{B}$ is a σ-algebra. Therefore, $\cup_{i=1}^{\infty}E_i\in\mathfrak{A}$.

   (c) Suppose that $E\in\mathfrak{A}$, i.e., $(E)_x\in\mathcal{B}$, then

   $$\begin{aligned}(E^c)_x&=\{y \mid (x,y)\in E^c\}\\&=\{y \mid (x,y)\notin E\}\\&=(E_x)^c\in\mathcal{B}\end{aligned}$$

which implies $E^c \in \mathfrak{A}$.

2. For any $A \times B \in \mathcal{A} \times \mathcal{B}$, since $(A \times B)_x = B \in \mathcal{B}$, we imply $(A \times B) \in \mathfrak{A}$.

In conclusion, $\mathfrak{A} = \mathcal{A} \otimes \mathcal{B}$. For all $E \in \mathcal{A} \otimes \mathcal{B}$, we imply $E \in \mathfrak{A}$, i.e., $E_x \in \mathcal{B}$. ∎

Proposition 13.3 Sippose that $f : X \times Y \to [-\infty, \infty]$ is measurable. (i.e., $f^{-1}((a,\infty]) \in \mathcal{A} \otimes \mathcal{B}$), then the maps

$$\begin{cases} f_x : & Y \to [-\infty,\infty] \\ \text{with} & f_x(y) := f(x,y) \end{cases}, \qquad \begin{cases} f_y : & X \to [-\infty,\infty] \\ \text{with} & f_y(x) := f(x,y) \end{cases}$$

are measurable. More precisely, $f_x^{-1}((a,\infty]) \in \mathcal{B}$ and $f_y^{-1}((a,\infty]) \in \mathcal{A}$.

Proof.

$$\begin{aligned} f_x^{-1}((a,\infty]) &= \{y \in Y \mid f_x(y) \in (a,\infty]\} \\ &= \{y \in Y \mid f(x,y) > a\} \\ &= \{(u,y) \in X \times Y \mid f(u,y) > a\}_x \\ &= (f^{-1}((a,\infty]))_x \in \mathcal{B} \end{aligned}$$

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