12.5. Wednesday for MAT3006

▪ **Example 12.7** Compute the integral

$$L = \lim_{n\to\infty}\int_0^1 \frac{nx\log(x)}{1+n^2x^2}\,\mathrm{d}x.$$

Let $f_n(x) = \frac{nx\log(x)}{1+n^2x^2}\mathcal{X}_{(0,1]}$, which is continuous on $[0,1]$, i.e., integrable on $[0,1]$. The goal is to show $L = 0$.

- Note that $f_n(x)\to 0, \forall x\in[0,1]$ pointwisely, as $n\to\infty$.
- Note that $t/(1+t^2)\le \frac{1}{2}, \forall t\ge 0$. Take $t = nx$, we imply

$$|f_n(x)| \le \frac{1}{2}|\log(x)|\mathcal{X}_{(0,1]}$$

We claim that $\frac{1}{2}|\log(x)|\mathcal{X}_{(0,1]} := -\frac{1}{2}\log(x)\mathcal{X}_{(0,1]}$ is integrable: by MCT I,

$$\int -\frac{1}{2}\log(x)\mathcal{X}_{(0,1]}\,\mathrm{d}m = \lim_{n\to\infty}\int_{1/n}^1 -\frac{1}{2}\log(x)\,\mathrm{d}x = \frac{1}{2} < \infty.$$

Therefore, the DCT applies, and

$$\lim_{n\to\infty}\int_0^1 \frac{nx\log(x)}{1+n^2x^2}\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty}\frac{nx\log(x)}{1+n^2x^2}\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0$$

However, $f_n(x)$ does not converge to $f(x)\equiv 0$ uniformly on $[0,1]$:

$$\sup_{0\le x\le 1}|f_n(x)-0| \ge |f_n(1/n)-0| = \frac{1}{2}\log(n)\to\infty, \text{as } n\to\infty$$

Therefore, we cannot switch integral symbol and limit by using the tools in MAT2006. ▪

Proposition 12.5 Suppose that $f(x)$ is a proper Riemann integrable function on $[a,b]$. Then $f(x)$ is Lebesgue integrable on $[a,b]$ with

$$\int_{[a,b]} f\,\mathrm{d}m = \int_a^b f(x)\,\mathrm{d}x.$$

Proof. Since f is properly Riemann inregrable, we imply $f(x)$ is bounded on $[a,b]$, i.e., $|f(x)| \leq K, \forall x \in [a,b]$. Construct the Riemann lower and upper functions with 2^n equal subintervals, denoted as ϕ_n, ψ_n, which follows that

- $\phi_n(x) \leq f(x) \leq \psi_n(x), \forall n$
- $\phi_n(x)$ is monotone increasing
- $\psi_n(x)$ is monotone decreasing

Now apply bounded convergence theorem on $\psi_n - \phi_n$:

- $|\psi_n(x) - \phi_n(x)| \leq 2K$ on $[a,b]$
- $\psi_n - \phi_n \to \psi - \phi$

which implies

$$\begin{aligned}
\int |\psi - \phi| \, \mathrm{d}m &= \int \psi - \phi \, \mathrm{d}m \\
&= \lim_{n\to\infty} \int \psi_n - \phi_n \, \mathrm{d}m = \lim_{n\to\infty} \int \psi_n \, \mathrm{d}m - \lim_{n\to\infty} \int \phi_n \, \mathrm{d}m \\
&= \text{Riemann Upper Sum} - \text{Riemann Lower Sum} \\
&= 0
\end{aligned}$$

Therefore, $\int |\psi - \phi| \, \mathrm{d}m = 0$ implies $\psi(x) = \phi(x)$ a.e. By sandwich theorem,

$$\psi(x) = f(x) = \phi(x) \text{ a.e.}$$

Therefore,

$$\int f \, \mathrm{d}m = \int \phi \, \mathrm{d}m = \lim_{n\to\infty} \int \phi_n \, \mathrm{d}m = \int_a^b f(x) \, \mathrm{d}x$$

where the second equality is by MCT II.

■

R The improper Riemann integrable functions $f(x)$ is not necessarily Lebesgue integrable. However, if we assume $f(x) \geq 0$, then $f(x)$ is improper Riemann integrable implies $f(x)$ is Lebesgue integrable, with the same integral value.

Proof Outline. Suppose $f(x)$ is improper Riemann integrable on $[a,b]$, where $a,b \in \mathbb{R} \cup \{\pm\infty\}$.

- Construct $f_n = f\mathcal{X}_{[a_n,b_n]}$, with $[a_n,b_n] \subseteq [a_{n+1},b_{n+1}] \subseteq \cdots \subseteq [a,b]$.
- By previous proposition, f_n is proper Riemann integrable implies f_n is Lebesgue integrable.
- Then we apply the MCT I to $\{f_n\}$.

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Theorem 12.4 — **Continous parameter DCT.** Let $I, J \subseteq \mathbb{R}$ be intervals, and $f : I \times J \to \mathbb{R}$ be such that

1. for fixed $y \in J$, the function $f(x) := f(x,y)$ is an integrable function over I.
2. for fixed $y \in J$,
$$\lim_{y' \to y} f(x,y') = f(x,y)$$
for almost all $x \in I$
3. There exists integrable $g(x)$ (do not depend on y) such that for all $y \in J$,
$$|f(x,y)| \le g(x)$$
for almost all $x \in I$.

As a result,
$$F(y) = \int_I f(x,y)\,\mathrm{d}x$$
is a continuous function on J.

Proof. Let $\{y_n\}$ be a sequence on J such that $y_n \to y$. It suffices to show $F(y_n) \to F(y)$.

Construct $f_n(x) = f(x,y_n)$, which follows that

- $f_n(x)$ is integrable for all n (by hypothesis (1)) (why check integrable)
- $|f_n(x)| \le g(x)$ a.e. for all n, and $g(x)$ is integrable (by hypothesis (3))

- By hypothesis (2),

$$\lim_{n\to\infty} f_n(x) = f(x,y)$$

Therefore, the DCT applies, and

$$\lim_{n\to\infty} \int_I f_n(x,y_n)\,\mathrm{d}m = \int \lim_{n\to\infty} f_n(x,y_n)\,\mathrm{d}m = \int_I f(x,y)\,\mathrm{d}m$$

Or equivalently,

$$\lim_{n\to\infty} F(y_n) = F(y)$$

■

■ **Example 12.8** Consider $f(x,y) = e^{-x}x^{y-1}$ with $I \times J = (0,\infty) \times [m,M]$, where $0 < m < M < \infty$. We will study the integral

$$\Gamma(y) = \int_0^\infty e^{-x}x^{y-1}\,\mathrm{d}x$$

We check the hypothesis in the Theorem (12.4):

1. For fixed $k \in [m,M]$, $f(x,y)$ is indeed integrable on $(0,\infty)$:

$$\left(e^{-x}x^{k-1}\right)\mathcal{X}_{(0,\infty)} \le 1 \cdot x^{k-1}\mathcal{X}_{(0,K]} + 10e^{-x/2}\mathcal{X}_{[K,\infty)}$$

   where K is a sufficiently large number in $(0,\infty)$.

2. The hypothesis (2) follows directly from the contiuity of $f(x,y)$
3.

$$\begin{aligned}|f(x,y)| &\le e^{-x}x^{m-1}\mathcal{X}_{[0,1]} + e^{-x}x^{M-1}\mathcal{X}_{(1,\infty)} \\ &\le x^{m-1}\mathcal{X}_{[0,1]} + e^{-x}x^{M-1}\mathcal{X}_{(1,\infty)}\end{aligned}$$

   Here $x^{m-1}\mathcal{X}_{[0,1]}$ is integrable. Following the similar argument in (1), we imply $e^{-x}x^{M-1}\mathcal{X}_{(1,\infty)}$ is integrable as well.

Therefore, $\Gamma(y)$ is continuous for any $m \leq y \leq M$. Since the choice of $0 < m < M < \infty$ is arbitrary, we imply $T(y)$ is continous on $(0, \infty)$.

In the next lecture we wish to show that

$$F'(y) = \int_I \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$$