12.2. Monday for MAT3006

12.2.1. Remarks on MCT

▪ **Example 12.2** The MCT can help us to compute the integral

$$\lim_{n\to\infty}\int_0^{n\pi}\cos\left(\frac{x}{2n}\right)xe^{-x^2}\,\mathrm{d}x$$

Construct $f_n(x)=\cos\left(\frac{x}{2n}\right)xe^{-x^2}\mathcal{X}_{[0,n\pi]}$.

- Since $\cos(x/2n)<\cos(x/2(n+1))$ for any $x\in[0,n\pi]$, we imply f_n is monotone increasing with n
- $f_n(x)$ is integrable for all n.
- f_n converges pointwise to $xe^{-x^2}\mathcal{X}_{[0,\infty)}$

Therefore, MCT I applies and

$$\lim_{n\to\infty}\int_0^{n\pi}\cos\left(\frac{x}{2n}\right)xe^{-x^2}\,\mathrm{d}x=\int\left(\lim_{n\to\infty}f_n\right)\mathrm{d}m$$

with

$$\lim_{n\to\infty}f_n=xe^{-x^2}\mathcal{X}_{[0,\infty)}.$$

Moreover,

$$\int\left(\lim_{n\to\infty}f_n\right)\mathrm{d}m=\lim_{m\to\infty}\int_0^m xe^{-x^2}\,\mathrm{d}x \tag{12.1a}$$

$$=\int_0^\infty xe^{-x^2}\,\mathrm{d}x \tag{12.1b}$$

$$=\frac{1}{2} \tag{12.1c}$$

where (12.1a) is by applying MCT I with $g_m(x)=xe^{-x^2}\mathcal{X}_{[0,m]}$. ▪

Then we discuss the Lebesgue integral for series:

Corollary 12.3 [Lebesgue Series Theorem] Let $\{f_n\}$ be a series of measurable functions such that

$$\sum_{n=1}^{\infty}\int |f_n|\,\mathrm{d}m < \infty,$$

then $\sum_{n=1}^{k} f_n$ converges to an integrable function $f = \sum_{n=1}^{\infty} f_n$ a.e., with

$$\int f\,\mathrm{d}m = \sum_{n=1}^{\infty}\int f_n\,\mathrm{d}m$$

Proof. • For each f_n, consider

$$f_n = f_n^+ - f_n^-, \text{ where } f_n^+, f_n^- \text{ are nonnegative.}$$

By proposition (11.6),

$$\int \sum_{n=1}^{\infty} f_n^+\,\mathrm{d}m = \sum_{n=1}^{\infty}\int f_n^+\,\mathrm{d}m \le \sum_{n=1}^{\infty}\int |f_n|\,\mathrm{d}m < \infty.$$

Therefore, $f^+ := \sum_{n=1}^{\infty} f_n^+ = \lim_{k\to\infty}\sum_{n=1}^{k} f_n^+$ is integrable. The same follows by replacing f^+ with f^-. By corollary (9.6), $f^+(x), f^-(x) < \infty, \forall x \in U$, where U^c is null.

• Therefore, construct

$$f(x) = \begin{cases} f^+(x) - f^-(x), & x \in U \\ 0, & x \in U^c \end{cases}$$

Moreover, for $x \in U$,

$$\begin{aligned} f(x) &= \left(\lim_{k\to\infty}\sum_{n=1}^{k} f_n^+(x)\right) - \left(\lim_{k\to\infty}\sum_{n=1}^{k} f_n^-(x)\right) \\ &= \lim_{k\to\infty}\left(\sum_{n=1}^{k} f_n^+(x) - \sum_{n=1}^{k} f_n^-(x)\right) \\ &= \lim_{k\to\infty}\left[\sum_{n=1}^{k}(f_n^+(x) - f_n^-(x))\right] \\ &= \sum_{n=1}^{\infty} f_n(x) \end{aligned}$$

where the first equality is because that both terms are finite.

- It follows that

$$\int f \,\mathrm{d}m = \int f^+ \,\mathrm{d}m - \int f^- \,\mathrm{d}m \tag{12.2a}$$

$$= \int \sum_{n=1}^{\infty} f_n^+ \,\mathrm{d}m - \int \sum_{n=1}^{\infty} f_n^- \,\mathrm{d}m \tag{12.2b}$$

$$= \left(\sum_{n=1}^{\infty} \int f_n^+ \,\mathrm{d}m\right) - \left(\sum_{n=1}^{\infty} \int f_n^- \,\mathrm{d}m\right) \tag{12.2c}$$

$$= \sum_{n=1}^{\infty} \left(\int f_n^+ \,\mathrm{d}m - \int f_n^- \,\mathrm{d}m\right) \tag{12.2d}$$

$$= \sum_{n=1}^{\infty} \int f_n \,\mathrm{d}m \tag{12.2e}$$

where (12.2a),(12.2d) is because that summation/subtraction between series holds when these series are finite; (12.2c) is by proposition (11.6); (12.2e) is by definition of f_n.

■

■ **Example 12.3** Compute the integral

$$\int_0^1 e^{-x} x^{\alpha-1} \,\mathrm{d}x, \ \alpha > 0.$$

- Construct $f_n(x) = (-1)^n \frac{x^{\alpha+n-1}}{n!} \mathcal{X}_{(0,1]}, n \geq 0$, and

$$\sum_{n=0}^{N} f_n(x) \to e^{-x} x^{\alpha-1}, \text{ pointwisely}, x \in (0,1].$$

By applying MCT I,

$$\int |f_n| \,\mathrm{d}m = \frac{1}{(\alpha+n)n!}$$

Therefore,

$$\sum_{n=0}^{\infty} \int |f_n| \,\mathrm{d}m = \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)n!} < \infty$$

- Applying the Lebesgue Series Theorem,

$$\int_0^1 e^{-x}x^{\alpha-1}\,\mathrm{d}x = \int_0^1 (\sum_{n=0}^{\infty} f_n)\,\mathrm{d}m = \sum_{n=0}^{\infty}\int f_n\,\mathrm{d}m = \sum_{n=0}^{\infty}\frac{(-1)^n}{(\alpha+n)n!}$$

R It's essential to have $\sum\int|f|\,\mathrm{d}m<\infty$ rather than $\sum\int f_n\,\mathrm{d}m<\infty$ in the Lebesgue Series Theorem. For example, let

$$f_n = \frac{(-1)^{n+1}}{(n+1)}\mathcal{X}_{[n,n+1)} \implies \sum_{n=1}^{\infty}\int f_n\,\mathrm{d}m = \log(2) < \infty$$

However, $f := \sum f_n$ is not integrable.

12.2.2. Dominated Convergence Theorem

Theorem 12.2 Let $\{f_n\}$ be a sequence of measruable functions such that $|f_n|\le g$ a.e., and g is integrable. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., then

1. f is integrable,
2. $$\int f\,\mathrm{d}m = \lim_{n\to\infty}\int f_n\,\mathrm{d}m$$

Proof.
- Observe that

$$|f_n|\le g \implies \lim_{n\to\infty}|f_n|\le g \implies |f|\le g$$

By comparison test, g is integrable implies $|f|$ is integrable, and further f is integrable.

- Consider the sequence of non-negative functions $\{g-f_n\}_{n\in\mathbb{N}}$ and $\{g+f_n\}_{n\in\mathbb{N}}$.

By Fatou's Lemma,

$$\begin{aligned}\liminf_{n\to\infty}\int (g-f_n)\,\mathrm{d}m &\geq \int \liminf_{n\to\infty}(g-f_n)\,\mathrm{d}m\\ &= \int (g-f)\,\mathrm{d}m\\ &= \int g\,\mathrm{d}m - \int f\,\mathrm{d}m\end{aligned}$$

which follows that

$$\int g\,\mathrm{d}m - \limsup_{n\to\infty}\int f_n\,\mathrm{d}m \geq \int g\,\mathrm{d}m - \int f\,\mathrm{d}m$$

i.e.,

$$\int f\,\mathrm{d}m \geq \limsup_{n\to\infty}\int f_n\,\mathrm{d}m$$

- Similarly,

$$\liminf_{n\to\infty}(g+f_n)\,\mathrm{d}m \geq \int \liminf_{n\to\infty}(g+f_n)\,\mathrm{d}m = \int g\,\mathrm{d}m + \int f\,\mathrm{d}m$$

which implies

$$\liminf_{n\to\infty}\int f_n\,\mathrm{d}m \geq \int f\,\mathrm{d}m$$

As a result,

$$\limsup_{n\to\infty}\int f_n\,\mathrm{d}m \leq \int f\,\mathrm{d}m \leq \liminf_{n\to\infty}\int f_n\,\mathrm{d}m,$$

which implies

$$\int f\,\mathrm{d}m = \lim_{n}\int f_n\,\mathrm{d}m$$

■

Corollary 12.4 [Bounded Convergence Theorem] Suppose that $E \in \mathcal{M}$ be such that $m(E) < \infty$. If

- $|f_n(x)| \leq K < \infty$ for any $x \in E, n \in \mathbb{N}$

- $f_n \to f$ a.e. in E,

then f is integrable in E with

$$\int_E f\,dm = \lim_{n\to\infty} \int f_n\,dm$$

Proof. Take $g = K\mathcal{X}_E$ in DCT. ■

Proposition 12.2 Every Riemann integrable function f on $[a,b]$ is Lebesgue integrable, without the condition that f is continuous a.e.

Proof. Since f is Riemann integrable, we imply f is bounded. We construct the Riemann lower abd upper functions with 2^n equal intervals, denoted as $\{\phi_n\}$ and $\{\psi_n\}$, which follows that

- ϕ_n is monotone increasing; ψ_n is monotone decreasing;
- $\phi_n \le f \le \psi_n$, and

$$\lim_{n\to\infty} \int_{[a,b]} \phi_n = \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_{[a,b]} \psi_n.$$

Construct $g = \sup_n \phi_n$ and $h = \inf_n \psi_n$. Now we can apply the bounded convergence theorem:

- ϕ_n is bounded on $[a,b]$
- $\phi_n \to g$ on $[a,b]$

which implies g is Lebesgue integrable on $[a,b]$, with

$$\int_{[a,b]} g\,dm = \lim_{n\to\infty} \int_{[a,b]} \phi_n = \int_a^b f(x)\,dx.$$

Similarly, h is Lebesgue integrable, with

$$\int_{[a,b]} h\,dm = \lim_{n\to\infty} \int_{[a,b]} \psi_n = \int_a^b f(x)\,dx.$$

Moreover, $g \leq f \leq h$, and

$$\int_{[a,b]} (h-g)\,\mathrm{d}m = \int_{[a,b]} h\,\mathrm{d}m - \int_{[a,b]} g\,\mathrm{d}m = \int_a^b f(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x = 0,$$

which implies $h = g$ a.e., and further $f = g$ a.e., which implies

$$\int_{[a,b]} f\,\mathrm{d}m = \int_{[a,b]} g\,\mathrm{d}m = \int_a^b f(x)\,\mathrm{d}x.$$

■

R However, an improper Riemann integral does not necessarily has the corresponding Lebesgue integral:

$$f(x) = \sum_{n=1}^{\infty} (-1)^n n \cdot \mathcal{X}_{(1/(n+1),1/n]}, \ x \in [0,1]$$

In this case, f is Riemann integrable but not Lebesgue integrable.