## 11.5. Wednesday for MAT3006

**Proposition 11.11** — **Linearity.** If $f, g$ are both integrable, then $f+g$ and $\alpha f$ are integrable with

$$\int (f+g)\,\mathrm{d}m = \int f\,\mathrm{d}m + \int g\,\mathrm{d}m$$
$$\int \alpha f\,\mathrm{d}m = \alpha \int f\,\mathrm{d}m \quad \alpha \in \mathbb{R}$$

*Proof.* 1. Construct

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-) \implies (f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

Since both sides for the equality above is non-negative, we do the Lebesgue integral both sides:

$$\int ((f+g)^+ + f^- + g^-)\,\mathrm{d}m = \int ((f+g)^- + f^+ + g^+)\,\mathrm{d}m.$$

Due to the linearity of Lebesgue integral for non-negative functions,

$$\int (f+g)^+\,\mathrm{d}m + \int f^-\,\mathrm{d}m + \int g^-\,\mathrm{d}m = \int (f+g)^-\,\mathrm{d}m + \int f^+\,\mathrm{d}m + \int g^+\,\mathrm{d}m$$

i.e.,

$$\int (f+g)\,\mathrm{d}m = \int f\,\mathrm{d}m + \int g\,\mathrm{d}m$$

2. Assume $\alpha < 0$. Then

$$\begin{aligned}
\int (\alpha f)\,\mathrm{d}m &:= \int (\alpha f)^+\,\mathrm{d}m - \int (\alpha f)^-\,\mathrm{d}m \\
&= \int (-\alpha) f^-\,\mathrm{d}m - \int (-\alpha) f^+\,\mathrm{d}m \\
&= (-\alpha)\int f^-\,\mathrm{d}m - (-\alpha)\int f^+\,\mathrm{d}m \\
&= \alpha\left(\int f^+\,\mathrm{d}m - \int f^-\,\mathrm{d}m\right) \\
&= \alpha \int f\,\mathrm{d}m
\end{aligned}$$

The proof for the case $\alpha \geq 0$ follows similarly.

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### 11.5.1. Properties of Lebesgue Integrable Functions

**Corollary 11.2** Suppose that $f, g$ are integrable, then

1. If $f \leq g$, then $\int f \, \mathrm{d}m \leq \int g \, \mathrm{d}m$
2. If $f = g$ a.e., then $\int f \, \mathrm{d}m = \int g \, \mathrm{d}m$

*Proof.* 1. Since $g - f \geq 0$, $\int (g-f) \, \mathrm{d}m \geq \int 0 \, \mathrm{d}m = 0$. By linearity, $\int g \, \mathrm{d}m - \int f \, \mathrm{d}m \geq 0$, i.e.,

$$\int g \, \mathrm{d}m \geq \int f \, \mathrm{d}m.$$

2. The proof follows similarly as in proposition (11.4). In detail, let $U = \{x \mid f(x) = g(x)\}$, then $m(U) = 0$. It follows that

$$\int f \mathcal{X}_{U^C} dm = \int f^+ \mathcal{X}_{U^c} dm + \int f^- \mathcal{X}_{U^c} dm = 0$$

Similarly, $\int g \mathcal{X}_{U^c} dm = 0$. Therefore,

$$\begin{aligned}
\int f \, \mathrm{d}m &= \int f \mathcal{X}_U \, \mathrm{d}m + \int f \mathcal{X}_{U^c} \, \mathrm{d}m \\
&= \int g \mathcal{X}_U \, \mathrm{d}m \\
&= \int g \mathcal{X}_U \, \mathrm{d}m + \int g \mathcal{X}_{U^c} \, \mathrm{d}m \\
&= \int g \, \mathrm{d}m
\end{aligned}$$

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1. Consider the set of integrable functions, say $\mathcal{T} = \{f : \mathbb{R} \to [-\infty, \infty], \text{ integrable}\}$, which is a vector space if we define $0_{\mathcal{T}} :=$ zero function.

We can define a "norm" on $f \in \mathcal{T}$ by

$$\|f\| = \int |f| \, \mathrm{d}m$$

then $\|\alpha f\| = |\alpha| \|f\|$ and $\|f + g\| \le \|f\| + \|g\|$.

Unfortunately, we should keep in mind that $\mathcal{T}$ is not a normed space, since there exists $f \neq 0_{\mathcal{T}}$ such that $\|f\| = 0$, e.g., $f = \mathcal{X}_{\mathbb{Q}}$.

2. To remedy this, define the equivalence relation on $\mathcal{T}$: $f \sim g$ if $f = g$ a.e. The equivalence classes of $\mathcal{T}$ under $\sim$ are of the form $[f] := \{g : g \sim f\}$. Denote the collection of equivalence classes as $L^1(\mathbb{R}) := \mathcal{T}/\sim$.

   (a) It's clear that $L^1(\mathbb{R})$ has a vector space structure

   $$[f] + [g] = [f + g]$$
   $$\alpha[f] = [\alpha f]$$

   (b) The space $L^1(\mathbb{R})$ can be viewed as a quotient space defined in linear algebra. Consider a vector subspace $\mathcal{N}$ of $\mathcal{T}$ defined by

   $$\mathcal{N} := \{g \in \mathcal{T} \mid g = 0 \text{ a.e.}\}$$

   then $\mathcal{T}/\sim = \mathcal{T}/\mathcal{N}$.

   (c) We define a norm on $L^1(\mathbb{R})$ by $\|[f]\| = \int |f| \, \mathrm{d}m$, which is truly a norm:

   $$\|\alpha[f]\| = |\alpha| \|[f]\|$$
   $$\|[f] + [g]\| \le \|[f]\| + \|[g]\|$$
   $$\|[f]\| = 0 \iff \int |f| \, \mathrm{d}m = 0 \iff f = 0 \text{ a.e.} \iff [f] = 0_{L^1(\mathbb{R})}$$

   Similarly, we can study $L^2(\mathbb{R}), \dots, L^p(\mathbb{R})$, e.g., for $L^2(\mathbb{R}) = \{f : \mathbb{R} \to [-\infty, \infty] \mid \int |f|^2 \, \mathrm{d}m < \infty\}/\mathcal{N}$, define the norm

   $$\|f\|_2 = \left(\int |f|^2 \, \mathrm{d}m\right)^{1/2}$$

■ **Example 11.10** There exist some improper Riemann integrable functions that are not Lebesgue integrable: Consider $f = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \mathcal{X}_{[k,k+1)}$, then the improper Riemann integral gives

$$\int_0^{\infty} f(x)\,dx = \log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

However, $f$ is not Lebesgue integrable. Suppose on the contrary that it is , then $|f|$ is integrable:

$$|f| = \sum_{k=0}^{\infty} \frac{1}{k+1} \mathcal{X}_{[k,k+1)}$$

However,

$$\int |f|\,dm = \lim_{n\to\infty} \sum_{k=0}^{n} \int \left( \frac{1}{k+1} \mathcal{X}_{[k,k+1)} \right) dm$$
$$= \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty$$

We will also show that all the proper Riemann integrable functions are Lebesgue integrable (and the integrals have the same value) ■

**Theorem 11.3 — MCT II.** Let $\{f_n\}$ be a sequence of integrable functions such that

1. $f_n \le f_{n+1}$ a.e.
2. $\sup_n \int f_n\,dm < \infty$

Then $f_n$ converges to an integrable function $f$ a.e., and

$$\int f\,dm = \lim_{n\to\infty} \int f_n\,dm$$

*Proof.* Re-define $f_n$ by changing its values on a null set such that

1. $f_n(x) \in \mathbb{R}$, for any $x \in \mathbb{R}$
2. $f_n(x) \le f_{n+1}(x)$ for any $n \in \mathbb{R}, x \in \mathbb{R}$

Let $f(x) = \lim_{n\to\infty} f(x)$. Consider the sequence of functions $\{f_n - f_1\}_{n\in\mathbb{N}}$, then

1. $f_n - f_1 \ge 0$

2. $f_n - f_1$ is monotone increasing, integrable
3. $f_n - f_1 \to f - f_1$

Applying MCT I gives

$$\int (f - f_1) \, \mathrm{d}m = \lim_{n\to\infty} \int (f_n - f_1) \, \mathrm{d}m$$

Adding $\int f_1 \, \mathrm{d}m$ and applying the linearity of integrals, we obtain

$$\int (f - f_1) \, \mathrm{d}m + \int f_1 \, \mathrm{d}m = \lim_{n\to\infty} \int (f_n - f_1) \, \mathrm{d}m + \int f_1 \, \mathrm{d}m = \lim_{n\to\infty} \int f_n \, \mathrm{d}m$$

Here $\lim_{n\to\infty} \int f_n \, \mathrm{d}m$ exists as $\lim_{n\to\infty} \int f_n \, \mathrm{d}m = \sup_n \int f_n \, \mathrm{d}m < \infty$; and $\int (f - f_1) \, \mathrm{d}m + \int f_1 \, \mathrm{d}m$ is integrable since it equals $\lim_{n\to\infty} \int f_n \, \mathrm{d}m < \infty$.

Therefore,

$$\text{LHS} = \int f \, \mathrm{d}m = \text{RHS} = \lim_{n\to\infty} \int f_n \, \mathrm{d}m.$$

The proof is complete. ■