## 11.2. Monday for MAT3006

**Reviewing.** Compute the integration

$$\int_0^1 (1-x)^{-1/2}\,\mathrm{d}x$$

*Solution.* 1. Construct $g_n(x)=(1-x)^{-1/2}\mathcal{X}_{[0,1-1/n]}$, then $g_n$ is monotone increasing and $g_n(x)\to(1-x)^{-1/2}\mathcal{X}_{[0,1)}$ pointwisely.

2. By applying MCT I,

$$\int_{[0,1)} (1-x)^{-1/2}\,\mathrm{d}x = \lim_{n\to\infty}\int g_n\,\mathrm{d}m = 2.$$

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Question: How to comptue $\int_{[0,1]}(1-x)^{-1/2}\,\mathrm{d}x$?

Answer:

$$(1-x)^{-1/2}\mathcal{X}_{[0,1]} = (1-x)^{-1/2}\mathcal{X}_{[0,1)} + (1-x)^{-1/2}\mathcal{X}_{\{1\}}$$

which follows that

$$\int (1-x)^{-1/2}\mathcal{X}_{[0,1]}\,\mathrm{d}m = \int (1-x)^{-1/2}\mathcal{X}_{[0,1)}\,\mathrm{d}m + \int (1-x)^{-1/2}\mathcal{X}_{\{1\}}\,\mathrm{d}m \tag{11.1a}$$

$$= \int_{[0,1)} (1-x)^{-1/2}\,\mathrm{d}x + 0 \tag{11.1b}$$

where (11.1b) is because that $(1-x)^{-1/2}\mathcal{X}_{\{1\}}=0$ a.e.

### 11.2.1. Consequences of MCT I

**Proposition 11.4** If $f,g$ are measurable non-negative functions, and $f=g$ a.e., then

$$\int f\,\mathrm{d}m = \int g\,\mathrm{d}m$$

*Proof.* Let $U = \{x \in \mathbb{R} \mid f(x) = g(x)\}$, then

$$f = f \cdot \mathcal{X}_U + f \cdot \mathcal{X}_{U^c}$$

where $U^c$ is null. As a result,

$$\begin{aligned}
\int f \,\mathrm{d}m &= \int f\mathcal{X}_U + \int f\mathcal{X}_{U^c} && (11.2a)\\
&= \int g\mathcal{X}_U + 0 && (11.2b)\\
&= \int g\mathcal{X}_U + \int g\mathcal{X}_{U^c} && (11.2c)\\
&= \int g \,\mathrm{d}m && (11.2d)
\end{aligned}$$

where (11.2b) is because that $f \cdot \mathcal{X}_{U^c} = 0$ a.e., and $f \cdot \mathcal{X}_U = g \cdot \mathcal{X}_U$; (11.2c) is becasue that $g \cdot \mathcal{X}_{U^c} = 0$ a.e. ■

**Proposition 11.5 — Slight Generalization of MCT I.** Suppose that $f_n(x)$ are nonnegative measurable functions such that

1. $f_n$ is monotone increasing a.e.
2. $f_n(x) \to f(x)$ a.e.

then

$$\lim_{n\to\infty} \int f_n \,\mathrm{d}m = \int f \,\mathrm{d}m$$

*Proof.* Construct the set $V_n = \{x \mid f_n(x) \le f_{n+1}(x)\}$ and $V = \bigcap_{n=1}^{\infty} V_n$. Since $f_n(x)$ is monotone increasing a.e., we imply $m(V_n^c) = 0$, and $m(V^c) \le \sum_{n=1}^{\infty} m(V_n^c) = 0$.

1. Construct $\tilde{f}_n(x)$ as follows:

$$\tilde{f}_n(x) = \begin{cases} f_n(x), \text{ if } x \in V \\ 0, \text{ if } x \in V^c \end{cases}$$

As a result,

- $\tilde{f}_n$ is monotone increasing
- Define a function $g : \mathbb{R} \to [0, \infty]$ such that $\lim_{n\to\infty} \tilde{f}_n(x) = g_n(x)$.

Apply the MCT I gives

$$\lim_{n\to\infty}\int \tilde{f}_n \,\mathrm{d}m = \int g \,\mathrm{d}m \tag{11.3a}$$

2. Note that $\{x \mid \tilde{f}_n(x) \neq f_n(x)\} \subseteq V^c$, where $V^c$ is null. Therefore, $f_n = f$ a.e., which implies

$$\int \tilde{f}_n \,\mathrm{d}m = \int f_n \,\mathrm{d}m \tag{11.3b}$$

3. Consider $V' = \{x \mid \lim_{n\to\infty} f_n(x) = f(x)\}$, and $(V')^c$ is null by hyphothesis. For any $x \in V \cap V'$, we imply

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tilde{f}_n(x).$$

Since $(V \cap V')^c$ is null, we imply $\tilde{f}_n(x) \to f$ a.e. Note that $\tilde{f}_n(x) \to g$, we imply $g = f$ a.e., which follows that

$$\int g \,\mathrm{d}m = \int f \,\mathrm{d}m \tag{11.3c}$$

Combining (11.3a) to (11.3c), we conclude that

$$\lim_{n\to\infty}\int f_n \,\mathrm{d}m = \lim_{n\to\infty}\int \tilde{f}_n \,\mathrm{d}m = \int g \,\mathrm{d}m = \int f \,\mathrm{d}m$$

■

**Proposition 11.6** Let $\{f_k\}$ be non-negative measurable and

$$f := \sum_{k=1}^{\infty} f_k,$$

then

$$\int f \,\mathrm{d}m = \sum_{k=1}^{\infty}\int f_k \,\mathrm{d}m$$

*Proof.* Firstly, $\int f \,\mathrm{d}m$ is well-defined since $f = \lim_{n\to\infty}\sum_{k=1}^{n} f_k$ is measurable.

Secondly, take $g_n = \sum_{k=1}^{n} f_k$, which implies $g_n$ is monotone increasing and $g_n \to f$. Apply MCT I gives the desired result. ■

■ **Example 11.3** Consider

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} x^n, \quad x \in [0,1)$$

Take $f_k = \frac{(2k)!}{4^k (k!)^2} x^k$. Applying proposition (11.6) gives

$$\int_{[0,1)} (1-x)^{-1/2} \, dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(2n)!}{4^n \cdot (n!)^2} x^n \, dx$$

Or equivalently,

$$2 = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)(n+1)!}$$

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## 11.2.2. MCT II

We now extend our study to all measurable functions instead of non-negativity.

**Definition 11.4** [Lebesgue integrable] Let $f$ be a measurable function, then let

$$f^+(x) = \begin{cases} f(x), & \text{if } f(x) > 0 \\ 0, & \text{if } f(x) \leq 0 \end{cases} = f(x)\mathcal{X}_{f^{-1}((0,\infty])}$$

and

$$f^-(x) = \begin{cases} -f(x), & \text{if } f(x) \leq 0 \\ 0, & \text{if } f(x) > 0 \end{cases} = -f(x)\mathcal{X}_{f^{-1}([-\infty,0])}$$

As a result, $f^+$ and $f^-$ are both measurable.

Note that

- $f(x) = f^+(x) - f^-(x)$
- $|f|(x) = f^+(x) + f^-(x)$

Now we define the Lebesgue integral of $f$ as

$$\int f \,\mathrm{d}m = \int f^+ \,\mathrm{d}m - \int f^- \,\mathrm{d}m$$

We say $f$ is **Lebesgue integrable** if both $f^+$ and $f^-$ are integrable, i.e., $\int f^\pm \,\mathrm{d}m < \infty$

**Proposition 11.7**

1. If $f$ is measurable, then $f$ is integrable if and only if $|f|$ is integrable
2. If $f$ is measurable, and $|f| \leq g$ with $g$ integrable, then $f$ is also integrable

*Proof.*

1. If $f$ is integrable, then $\int f^+ \,\mathrm{d}m, \int f^- \,\mathrm{d}m < \infty$. As a result,

$$\int |f| \,\mathrm{d}m = \int (f^+ + f^-) \,\mathrm{d}m = \int f^+ \,\mathrm{d}m + \int f^- \,\mathrm{d}m < \infty.$$

For the reverse direction, if $|f|$ is integrable, then

$$\int |f| = \int f^+ + \int f^-$$

therefore $\int f^\pm < \infty$, and hence $f$ is interable.

2. Since $0 \leq |f| \leq g$, by proposition (9.8), $\int |f| \,\mathrm{d}m \leq \int g \,\mathrm{d}m < \infty$.

Therefore, $\int |f| \,\mathrm{d}m < \infty$, and hence $|f|$ is integrable, which implies $f$ is integrable.

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R If $|f| \leq g$, and $\int |f| \,\mathrm{d}m = \infty$, then by proposition (9.8), we imply $\int g \,\mathrm{d}m = \infty$.