10.5. Wednesday for MAT3006

Proposition 10.10 — **Fatou's Lemma.** Suppose $\{f_n\}$ is a sequence of measurable, non-negative functions.

$$\liminf_{n\to\infty}\int f_n\,\mathrm{d}m \ge \int \liminf_{n\to\infty}(f_n)\,\mathrm{d}m$$

Proof. Define

$$f(x)=\liminf_{n\to\infty} f_n(x)=\lim_{n\to\infty}\left(\inf_{k\ge n} f_k(x)\right):=\lim_{n\to\infty} g_n(x)$$

To study the integral $\int f\,\mathrm{d}m$, we will only focus on $f(x)$ on $E\subseteq\mathbb{R}$, where $f(x)>0,\forall x\in E$.

It suffices to show that $\int_E \phi\,\mathrm{d}m \le \lim_{n\to\infty}\inf\int_E f_n\,\mathrm{d}m$ for all simple ϕ satisfying $0\le\phi(x)\le f(x),\forall x\in E$. (Then taking supremum both sides leads to the desired result.)

1. Construct the simple function ϕ' on E such that

$$\phi'(x)=\begin{cases}\phi(x)-\varepsilon, & \text{if } \phi(x)>0\\ 0, & \text{if } \phi(x)=0\end{cases}$$

   in which we pick ε small enough such that $\phi(x)-\varepsilon\ge 0$.

   As a result, $\phi'<f,\forall x\in E$ (why?).

2. Note that $g_n(x)$ is monotone increasing with n, and therefore convergent to $f(x)$. Consider $A_n:=\{x\in E\mid \phi'(x)\le g_n(x)\}$, which follows that

   (a) $A_n\subseteq A_{n+1}$

   (b) $\cup_{n=1}^{\infty}A_n=E$ (We do need ϕ' is strictly less than f to obtain this condition).

   Therefore, for any $k\ge n$,

$$\int_{A_n}\phi'\,\mathrm{d}m\le\int_{A_n} g_n\,\mathrm{d}m\le\int_{A_n} f_k\,\mathrm{d}m,$$

   which implies $\int_{A_n}\phi'\,\mathrm{d}m\le\int_E f_k\,\mathrm{d}m$ since $f_k\mathcal{X}_{A_n}\le f_k\mathcal{X}_E$. Or equivalently,

$$\int_{A_n}\phi'\,\mathrm{d}m\le\inf_{k\ge n}\int_E f_k\,\mathrm{d}m \tag{10.2}$$

3. Taking limits $n\to\infty$ both sides for (10.2):

- For LHS, suppose that $\phi' = \sum_i \alpha_i \mathcal{X}_{c_i}$, then $\int_{A_n} \phi' \, \mathrm{d}m = \sum_i \alpha_i m(c_i \cap A_n)$, which follows that

$$\lim_{n\to\infty} \int_{A_n} \phi' \, \mathrm{d}m = \sum_i \alpha_i \lim_{n\to\infty} m(c_i \cap A_n) = \sum_i \alpha_i m(c_i) = \int_E \phi' \, \mathrm{d}m$$

- The limit of RHS equals $\lim_{n\to\infty} \inf \int_E f_n \, \mathrm{d}m$, and therefore

$$\int_E \phi' \, \mathrm{d}m \le \lim_{n\to\infty} \inf \int_E f_n \, \mathrm{d}m$$

Note that the goal is to show $\int_E \phi \, \mathrm{d}m \le \lim_{n\to\infty} \inf \int_E f_n \, \mathrm{d}m$, and therefore we need to evaluate ϕ' in terms of ϕ.

4. (a) Consider the case where $m(\phi^{-1}(0,\infty)) = P < \infty$, then

$$\int_E \phi' \, \mathrm{d}m = \int_E \phi \, \mathrm{d}m - \varepsilon \cdot P \le \lim_{n\to\infty} \inf \int_E f_n \, \mathrm{d}m,$$

for all small $\varepsilon > 0$. Then the desired result holds.

(b) Consider the case where $m(\phi^{-1}(0,\infty)) = \infty$, and we write the canonical form $\phi = \sum \alpha_i \mathcal{X}_{c_i}$ with $\alpha_i > 0$. Define $C = \cup_i c_i$ such that $m(c) = \infty$.

Construct the simple function $\phi' = a\mathcal{X}_C$, where $a := \frac{1}{2} \min\{\alpha_i\}$, which implies

- $\phi' \le \phi$
- $\int_E \phi' \, \mathrm{d}m = am(c) = \infty$, which follows that $\int_E \phi \, \mathrm{d}m = \infty$.

Our goal is to show $\lim_{n\to\infty} \inf \int_E f_n \, \mathrm{d}m = \infty$.

Consider $B_n = \{x \in E \mid g_n(x) > a\}$, then $\cup B_n = E, B_n \subseteq B_{n+1}$.

Observe the inequality

$$\int_{C\cap B_n} a \, \mathrm{d}m \le \int_{B_n} a \, \mathrm{d}m \le \int_{B_n} g_n \, \mathrm{d}m \le \inf_{k\ge n} \int_E f_n \, \mathrm{d}m$$

Taking $n \to \infty$ both sides. For LHS, by definition of B_n, the limit equals $\int_C a \, \mathrm{d}m = \int \phi' \, \mathrm{d}m = \infty$; and the limit of RHS equals to $\lim_{n\to\infty} \inf \int_E f_n \, \mathrm{d}m$, i.e.,

$$\lim_{n\to\infty} \inf \int_E f_n \, \mathrm{d}m = \infty$$

∎

Theorem 10.2 — **Monotone Convergence Theorem I.** Let $\{f_n\}$ be a sequence of non-negative measurable functions, with

- $f_n(x)$ being monotone increasing
- $f_n(x) \to f(x)$ pointwisely

Then we have

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}m = \int \left(\lim_{n\to\infty} f_n\right) \mathrm{d}m := \int f \, \mathrm{d}m$$

Proof.

- On the one hand, for all $n \in \mathbb{N}$, we have

$$f_n \le f \implies \int f_n \, \mathrm{d}m \le \int f \, \mathrm{d}m \implies \limsup_{n\to\infty} \int f_n \, \mathrm{d}m \le \int f \, \mathrm{d}m$$

- On the other hand, applying the Fatou's lemma,

$$\int f \, \mathrm{d}m := \int \left(\liminf_{n\to\infty} f_n\right) \mathrm{d}m \le \liminf_{n\to\infty} \int f_n \, \mathrm{d}m$$

  Togehter with the previous inequality, we imply

$$\limsup_{n\to\infty} \int f_n \, \mathrm{d}m \le \int f \, \mathrm{d}m \le \liminf_{n\to\infty} \int f_n$$

Therefore, all inequalities above are equalities, and the limit exists since limsup and liminf coincides. Moreover,

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}m = \int f \, \mathrm{d}m.$$

∎

From MCT I, the Lebesgue integral $\int f \, \mathrm{d}m$ can be computed as follows:

- Construct simple functions $\phi_n \le \phi_{n+1}$ with $\phi_n \to f$
- Evaluate $\int \phi_n \, \mathrm{d}m$ and then $\int f \, \mathrm{d}m = \lim_{n\to\infty} \int \phi_n \, \mathrm{d}m$

10.5.1. Consequences of MCT

Proposition 10.11 The Lebesgue integral is finitely addictive for measurable non-negative functions. In other words, suppose f, g are measurable and nonnegative, then

$$\int f \, dm + \int g \, dm = \int (f+g) \, dm$$

Proof. Suppose we have simple increasing functions $\{\phi_n\}$ and $\{\psi_n\}$ such that $\phi_n \to f$ and $\psi_n \to f$. Then

$$\begin{aligned} \int (f+g) \, dm &= \lim_{n\to\infty} \int (\phi_n + \psi_n) \, dm && (10.3a) \\ &= \lim_{n\to\infty} \int \phi_n \, dm + \lim_{n\to\infty} \int \psi_n \, dm && (10.3b) \\ &= \int f \, dm + \int g \, dm && (10.3c) \end{aligned}$$

where (10.3a) and (10.3c) is by applying MCT I; and (10.3b) is by definition of simple function. ■

> **Corollary 10.1** The Lebesgue integral is linear defined for measurable, nonnegative functions. In other words, suppose f, g are measurable and nonnegative, then
>
> $$\int (af + bg) \, dm = a \int f \, dm + b \int g \, dm,$$
>
> for any $a, b \geq 0$.

Proposition 10.12 The Lebesgue integral for non-negative continuous function on a bounded closed interval coincides with the Riemann integral. In other words, let f be a non-negative continuous function on $[a, b]$. then

$$\int_{[a,b]} f \, dm = \int_a^b f(x) \, dx.$$

We will extend this result into all proper Riemann integrable functions on $[a, b]$ soon.

Proof. Let ϕ_n be the simple function giving the Riemann lower sum of $f(x)$ with 2^n

equal subintervals:

$$\phi_n(x) = \sum_{k=1}^{2^n} \left(\min_{y \in \bar{I}_k} f(y)\right) \mathcal{X}_{I_k}, \text{ where } I_k = [a + (b-a)\frac{k-1}{2^n}, a + (b-a)\frac{k}{2^n}]$$

- $\phi_n(x) \geq 0$ is monotone increasing (that's the reason we should divde intervals into 2^n pieces instead of n pieces)
- $\phi_n(x) \to f(x)$ pointwisely: for any $x \in [a,b]$ and $\varepsilon > 0$, by (uniform) continuity of f, there exists $\delta > 0$ such that

$$|y - x| < \delta \implies |f(y) - f(x)| < \varepsilon.$$

Therefore, for sufficiently large n, we imply for any $x \in I_{k,n}$, $|I_{k,n}| < \delta$. As a result,

$$\left|\min_{y \in I_{k,n}} f(y) - f(x)\right| < \varepsilon.$$

Therefore,

$$\begin{aligned}
\int_{[a,b]} f \, \mathrm{d}m &= \lim_{n\to\infty} \int \phi_n \, \mathrm{d}m \\
&= \lim_{n\to\infty} \left[\text{Riemann lower integral of } \int_a^b f(x) \, \mathrm{d}x\right] \\
&= \int_a^b f(x) \, \mathrm{d}x
\end{aligned}$$

■

▪ **Example 10.5** The Lebesgue integral gives us an alternative way to compute improper integrals. Suppose that we want to compute the integral

$$\int_0^1 (1-x)^{-1/2} \, \mathrm{d}x.$$

1. The old method is that we know the integral

$$\int_0^{1-1/n} (1-x)^{-1/2} \text{ exists for any } n.$$

Then we extend the definition of Riemann integration by taking limit of n:

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^{1-1/n} (1-x)^{-1/2}$$

2. The Lebesgue integration does not require us to extend the definition. Consider

$$f_n(x) = (1-x)^{-1/2} \mathcal{X}_{[0,1-1/n]}$$

Then

- $f_n(x) \to f(x)$ on $[0,1)$
- $f_n(x)$ is monotone increasing

Therefore, by applying MCT I,

$$\int_0^1 (1-x)^{-1/2}\,\mathrm{d}x = \lim_{n\to\infty} \int f_n\,\mathrm{d}m = \lim_{n\to\infty} \int_0^{1-1/n} (1-x)^{-1/2}\,\mathrm{d}x.$$

∎