10.2. Monday for MAT3006

10.2.1. Remarks on Markov Inequality

Proposition 10.1 — **Markov Inequality.** Suppose that $f:\mathbb{R}\to[0,\infty]$ is measurable, then

$$m(f^{-1}[\lambda,\infty])\le\frac{1}{\lambda}\int f\,\mathrm{d}m,\ \forall\lambda>0$$

Proof. Define the function

$$g:=\lambda\mathcal{X}_{f^{-1}([\lambda,\infty])},$$

it follows that $g\le f$ globally. Applying proposition (9.8), we imply

$$\int g\,\mathrm{d}m\le\int f\,\mathrm{d}m\implies\lambda m(f^{-1}[\lambda,\infty])\le\int f\,\mathrm{d}m.$$

■

Corollary 10.1 If $f:\mathbb{R}\to[0,\infty]$ is integrable, and $\int f\,\mathrm{d}m=0$, then $f=0$ a.e.

Proof. Consider that for any $\lambda>0$,

$$0\le m(f^{-1}[\lambda,\infty])\le\frac{1}{\lambda}\int f\,\mathrm{d}m=0.$$

Therefore, $m(\{x\mid f(x)\ne0\})=m(f^{-1}(0,\infty])=0$. ■

10.2.2. Properties of Lebesgue Integration

In this lecture, we will show several lemmas, which is very useful during the proof of monotone convergence theorem.

Proposition 10.2 If $f:\mathbb{R}\to[0,\infty]$ is such that $f=0$ a.e., then $\int f\,\mathrm{d}m=0$.

Proof. Any simple function $\psi \leq f$ must be 0 almost everywhere:

$$\phi = \sum_i \alpha_i \mathcal{X}_{A_i}, \alpha_i > 0, \ \cup_i A_i \text{ is null.}$$

Direct computation of the Lebesgue integral for this simple function ψ gives

$$\int f \, \mathrm{d}m = \sum_i \alpha_i m(A_i) = 0,$$

where the last equality is because that for each i, the set A_i is null. ∎

(R) Given a non-negative integrable function f on a measurable set E, the integral $\int_E f \, \mathrm{d}m = 0$ if and only if $f = 0$ a.e. on E.

Proposition 10.3 If A, B are measurable, disjoint sets, then

$$\int_{A \cup B} f \, \mathrm{d}m = \int_A f \, \mathrm{d}m + \int_B f \, \mathrm{d}m$$

Proof. The key is to apply $f \cdot \mathcal{X}_{A \cup B} = f \cdot \mathcal{X}_A + f \cdot \mathcal{X}_B$ and

$$\int_E f \, \mathrm{d}m = \int f \cdot \mathcal{X}_E \, \mathrm{d}m, \text{ for any measurable } E.$$

∎

Proposition 10.4 If $f : \mathbb{R} \to [0, \infty]$ is measurable, then there exists an increasing sequence of simple functions $\{\phi_n\}$ such that $\phi_n(x) \to f(x)$ pointwise.

Proof. For each $n \in \mathbb{N}$, we divide the interval $[0, 2^n] \subseteq [0, \infty]$ into 2^{2n} subintervals of width 2^{-n}:

$$I_{k,n} = (k2^{-n}, (k+1)2^{-n}], \quad k = 0, 1, \dots, 2^{2n} - 1.$$

Let $J_n = (2^n, \infty]$ be the remaining part of the range of f, and define

$$E_{k,n} = f^{-1}(I_{k,n}), \quad F_n = f^{-1}(J_n).$$

Then the sequence of simple functions are given by:

$$\phi_n = \sum_{k=0}^{2^n-1} k \cdot 2^{-n} \chi_{E_{k,n}} + 2^n \chi_{F_n}.$$

■

Proposition 10.5 — Fatou's Lemma. Let $\{F_n\}$ be a sequence of non-negative measurable functions, then

$$\liminf_{n\to\infty} \int f_n \, \mathrm{d}m \geq \int \left(\liminf_{n\to\infty} f_n\right) \mathrm{d}m$$

(R) The inequality in the Fatou's lemma could be strict, e.g., consider $f_n(x) = (n+1)x^n$ on $[0,1]$.