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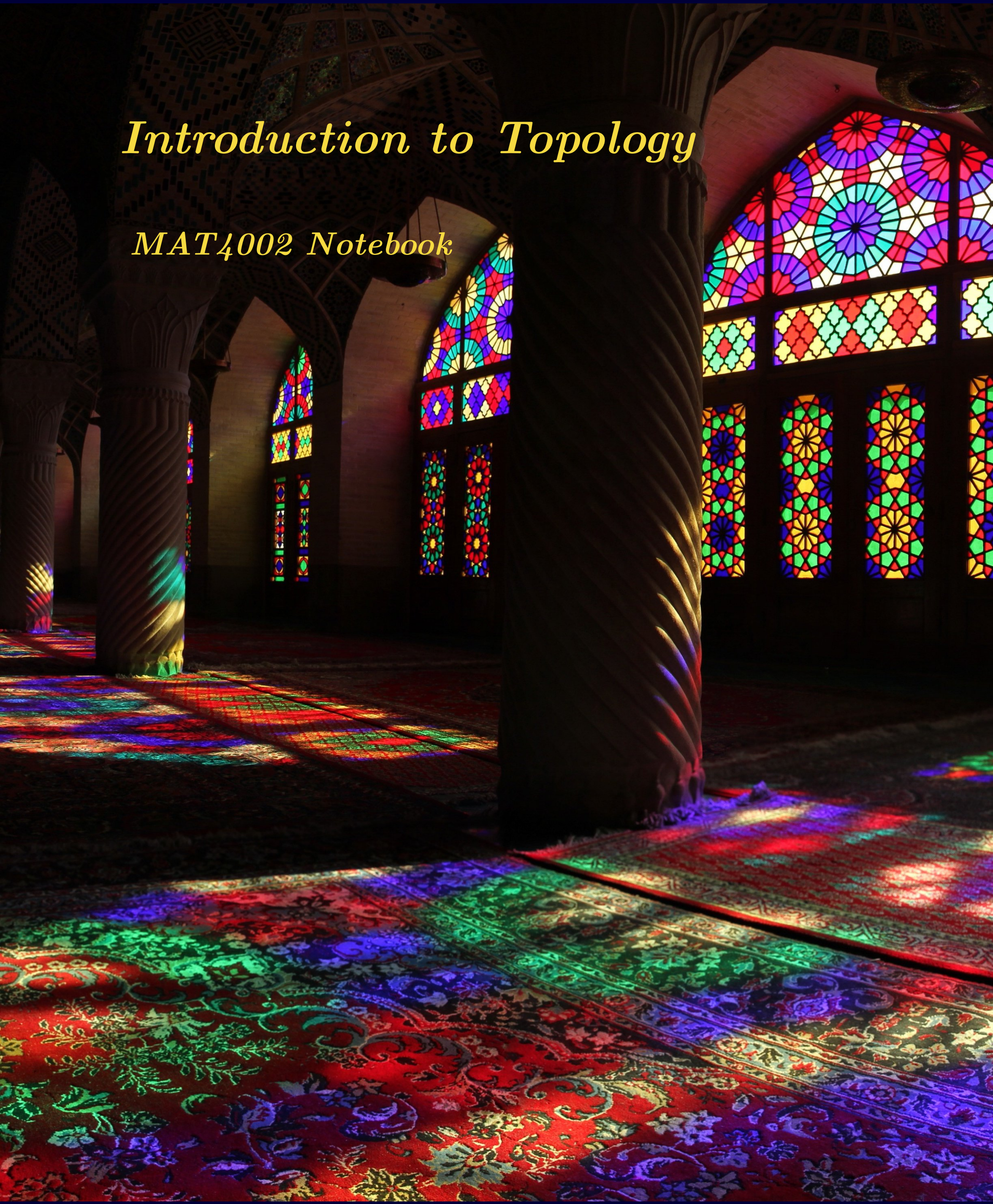

Introduction to Topology

MAT4002 Notebook

The First Edition

A FIRST COURSE

IN

TOPOLOGY

A FIRST COURSE

IN

TOPOLOGY

MAT4002 Notebook

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Acknowledgments

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Notations and Conventions

$(X, \mathcal{T})$	Topological space
$X \cong Y$	The space X is homeomorphic to space Y
$G \cong H$	The group G is isomorphic to group H
p_X	Project mapping
$X \times Y$	Product Topology
$X/\sim$	Quotient Topology related to the topologcial space X and the equivalence class $\sim$
S^n	The n-sphere $\{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\| = 1\}$
D^n	The n-disk $\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| \leq 1\}$
$E^\circ, \partial E, \overline{E}$	The interior, boundary, closure of E
$\mathbb{T}^2$	The torus in $\mathbb{R}^3$
Δ^n	The n-simplex
$i : A \hookrightarrow X$	Inclusion mapping from $A \subseteq X$ to X
$K = (V, \Sigma)$	(Abstract) Simplicial Complex
$\lvert K \rvert$	Topological realization of the simplicial complex K
$\langle X \mid R \rangle$	The presentation of a group
$H : f \overset{H}{\simeq} g$	f and g are homotopic, where H denotes the homotopy
$X \simeq Y$	The space X and Y are homotopy equivalent
$\pi_1(X, x)$	The fundamental group of X w.r.t. the basepoint $x \in X$
$E(K, b)$	The edge loop group of the space K w.r.t. the basepoint b
f_*	The induced homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ for $f : X \to Y$

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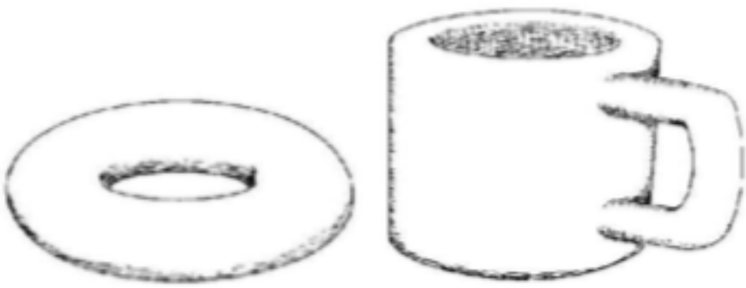
1.3. Monday for MAT4002

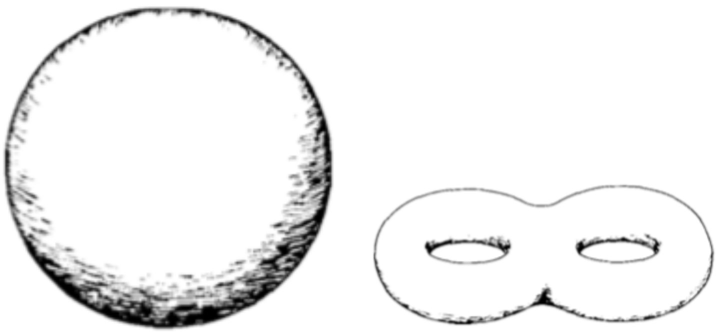
1.3.1. Introduction to Topology

We will study global properties of a geometric object, i.e., *the distrance between 2 points in an object is totally ignored*. For example, the objects shown below are essentially invariant under a certain kind of transformation:

Another example is that the coffee cup and the donut have the same topology:



However, the two objects below have the intrinsically different topologies:



In this course, we will study the phenomenon described above mathematically.

1.3.2. Metric Spaces

In order to ingnore about the distances, we need to learn about distances first.

Definition 1.7 [Metric Space] Metric space is a set X where one can measure distance between any two objects in X.

Specifically speaking, a metric space X is a non-empty set endowed with a function (distance function) $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x},\boldsymbol{y} \in X$ with equality iff $\boldsymbol{x} = \boldsymbol{y}$
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$ (triangular inequality)

Example 1.10 1. Let $X = \mathbb{R}^n$, with

$$d_2(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

$$d_\infty(\boldsymbol{x},\boldsymbol{y}) = \max_{i=1,\dots,n} |x_i - y_i|$$

2. Let X be any set, and define the discrete metric

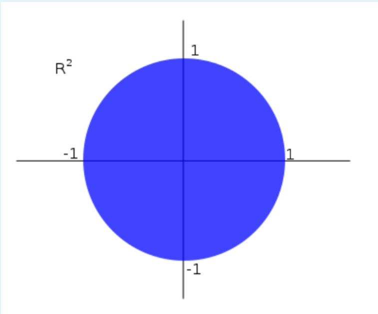
$$d(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Homework: Show that (1) and (2) defines a metric.

Definition 1.8 [Open Ball] An **open ball** of radius r centered at $\boldsymbol{x} \in X$ is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}$$

■ **Example 1.11** 1. The set $B_1(0,0)$ defines an open ball under the metric $(X = \mathbb{R}^2, d_2)$, or the metric $(X = \mathbb{R}^2, d_\infty)$. The corresponding diagram is shown below:



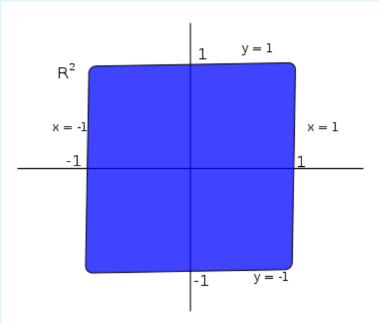


Figure 1.3: Left: under the metric $(X = \mathbb{R}^2, d_2)$; Right: under the metric $(X = \mathbb{R}^2, d_\infty)$

2. Under the metric $(X = \mathbb{R}^2,$ discrete metric$)$, the set $B_1(0,0)$ is one single point, also defines an open ball.

■

Definition 1.9 [Open Set] Let X be a metric space, $U \subseteq X$ is an open set in X if $\forall u \in U$, there exists $\epsilon_u > 0$ such that $B_{\epsilon_u}(u) \subseteq U$. ■

Definition 1.10 The **topology** induced from (X, d) is the collection of all open sets in (X, d), denoted as the symbol $\mathcal{T}$. ■

Proposition 1.5 All open balls $B_r(\boldsymbol{x})$ are open in (X, d).

Proof. Consider the example $X = \mathbb{R}$ with metric d_2. Therefore $B_r(x) = (x - r, x + r)$. Take $\boldsymbol{y} \in B_r(\boldsymbol{x})$ such that $d(\boldsymbol{x}, \boldsymbol{y}) = q < r$ and consider $B_{(r-q)/2}(\boldsymbol{y})$: for all $z \in B_{(r-q)/2}(\boldsymbol{y})$, we have

$$d(\boldsymbol{x}, \boldsymbol{z}) \le d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) < q + \frac{r - q}{2} < r,$$

which implies $\boldsymbol{z} \in B_r(x)$. ■

Proposition 1.6 Let $(X, \boldsymbol{d})$ be a metric space, and $\mathcal{T}$ is the topology induced from (X, d), then

1. let the set $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of (uncountable) open sets, i.e., $G_\alpha \in \mathcal{T}$,

then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha \in \mathcal{T}$.

2. let $G_1, \dots, G_n \in \mathcal{T}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$. The finite intersection of open sets is open.

Proof. 1. Take $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$, then $x \in G_\beta$ for some $\beta \in \mathcal{A}$. Since G_β is open, there exists $\epsilon_x > 0$ s.t.

$$B_{\epsilon_x}(x) \subseteq G_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} G_\alpha$$

2. Take $x \in \bigcap_{i=1}^n G_i$, i.e., $x \in G_i$ for $i = 1, \dots, n$, i.e., there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subseteq G_i$ for $i = 1, \dots, n$. Take $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$, which implies

$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq G_i, \forall i$$

which implies $B_\epsilon(x) \subseteq \bigcap_{i=1}^n G_i$

■

Exercise.

1. let $\mathcal{T}_2, \mathcal{T}_\infty$ be topologies induced from the metrices d_2, d_∞ in $\mathbb{R}^2$. Show that $J_2 = J_\infty$, i.e., every open set in $(\mathbb{R}^2, d_2)$ is open in $(\mathbb{R}^2, d_\infty)$, and every open set in $(\mathbb{R}^2, d_\infty)$ is open in $(\mathbb{R}_2, d_2)$.
2. Let $\mathcal{T}$ be the topology induced from the discrete metric (X, d_{discrete}). What is $\mathcal{T}$?

1.6. Wednesday for MAT4002

Reviewing.

- Metric Space (X,d)
- Open balls and open sets (note that the emoty set $\emptyset$ is open)
- Define the collection of open sets in X, say $\mathcal{T}$ is the topology.

Exercise.

1. Show that the $\mathcal{T}_2$ under $(X=\mathbb{R}^2, d_2)$ and $\mathcal{T}_\infty$ under $(X=\mathbb{R}^2, d_\infty)$ are the same.

   Ideas. Follow the procedure below:

   An open ball in d_2-metric is open in d_∞;

   Any open set in d_2-metric is open in d_∞;

   Switch d_2 and d_∞. ■

2. Describe the topology $\mathcal{T}_{\text{discrete}}$ under the metric space $(X=\mathbb{R}^2, d_{\text{discrete}})$.

   Outlines. Note that $\{x\} = B_{1/2}(x)$ is an open set.

   For any subset $W \subseteq \mathbb{R}^2$, $W = \bigcup_{w\in W}\{w\}$ is open.

   Therefore $\mathcal{T}_{\text{discrete}}$ is all subsets of $\mathbb{R}^2$. ■

1.6.1. Forget about metric

Next, we will try to define closedness, compactness, etc., without using the tool of metric:

Definition 1.18 [closed] A subset $V \subseteq X$ is **closed** if $X \setminus V$ is **open**.

Example 1.19 Under the metric space $(\mathbb{R}, d_1)$,

$$\mathbb{R}\setminus[b,a] = (a,\infty)\bigcup(-\infty,b) \text{ is open} \implies [b,a] \text{ is closed}$$

Proposition 1.14 Let X be a metric space.

1. $\emptyset, X$ is closed in X
2. If F_α is closed in X, so is $\bigcap_{\alpha\in A} F_\alpha$.
3. If $F_1,\dots,F_k$ is closed, so is $\bigcup_{i=1}^k F_i$.

Proof.
1. Note that X is open in X, which implies $\emptyset = X\setminus X$ is closed in X;

   Similarly, $\emptyset$ is open in X, which implies $X = X\setminus\emptyset$ is closed in X;
2. The set F_α is closed implies there exists open $U_\alpha \subseteq X$ such that $F_\alpha = X\setminus U_\alpha$. By De Morgan's Law,

$$\bigcap_{\alpha\in A} F_\alpha = \bigcap_{\alpha\in A}(X\setminus U_\alpha) = X\setminus(\bigcup_{\alpha\in A} U_\alpha).$$

   By part (a) in proposition (1.6), the set $\bigcup_{\alpha\in A} U_\alpha$ is openm which implies $\bigcap_{\alpha\in A} F_\alpha$ is closed.
3. The result follows from part (b) in proposition (1.6) by taking complements.

■

We illustrate examples where open set is used to define convergence and continuity.

1. Convergence of sequences:

   Definition 1.19 [Convergence] Let (X,d) be a metric space, then $\{x_n\}\to x$ means

$$\forall\varepsilon>0, \exists N \text{ such that } d(x_n,x)<\varepsilon, \forall n\ge N.$$

   We will study the convergence by using open sets instead of metric.

   Proposition 1.15 Let X be a metric space, then $\{x_n\}\to x$ if and only if for $\forall$ open set $U\ni x$, there exists N such that $x_n\in U$ for $\forall n\ge N$.

   Proof. Necessity: Since $U\ni x$ is open, there exists $\varepsilon>0$ such that $B_\varepsilon(x)\subseteq U$. Since $\{x_n\}\to x$, there exists N such that $d(x_n,x)<\varepsilon$, i.e., $x_n\in B_\varepsilon(x)\subseteq U$ for $\forall n\ge N$.

   Sufficiency: Let $\varepsilon>0$ be given. Take the open set $U=B_\varepsilon(x)\ni x$, then there exists N

such that $x_n \in U = B_\varepsilon(x)$ for $\forall n \geq N$, i.e., $d(x_n, x) < \varepsilon$, $\forall n \geq N$.

■

2. Continuity:

Definition 1.20 [Continuity] Let (X, d) and (Y, ρ) be given metric spaces. Then $f : X \to Y$ is continuous at $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon.$$

The function f is continuous on X if f is continous for all $x_0 \in X$.

We can get rid of metrics to study continuity:

Proposition 1.16 (a) The function f is continuous at x if and only if for all open $U \ni f(x)$, there exists $\delta > 0$ such that the set $B(x, \delta) \subseteq f^{-1}(U)$.

(b) The function f is continuous on X if and only if $f^{-1}(U)$ is open in X for each open set $U \subseteq Y$.

During the proof we will apply a small lemma:

Proposition 1.17 f is continuous at x if and only if for all $\{x_n\} \to x$, we have $\{f(x_n)\} \to f(x)$.

Proof. (a) *Necessity*:

Due to the openness of $U \ni f(x)$, there exists a ball $B(f(x), \varepsilon) \subseteq U$.

Due to the continuity of f at x, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$, which implies

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq U,$$

which implies $B(x, \delta) \subseteq f^{-1}(U)$.

Sufficiency:

Let $\{x_n\} \to x$. It suffices to show $\{f(x_n)\} \to f(x)$. For each open $U \ni f(x)$, by hypothesis, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$.

Since $\{x_n\} \to x$, there exists N such that

$$x_n \in B_\delta(x) \subseteq f^{-1}(U), \forall n \geq N \implies f(x_n) \in U, \forall n \geq N$$

Let $\varepsilon > 0$ be given, and then construct the $U = B_\varepsilon(f(x))$. The argument above shows that $f(x_n) \in B_\varepsilon(f(x))$ for $\forall n \geq N$, which implies $\rho(f(x_n), f(x)) < \varepsilon$, i.e., $\{f(x_n)\} \to f(x)$.

(b) For the forward direction, it suffices to show that each point x of $f^{-1}(U)$ is an interior point of $f^{-1}(U)$, which is shown by part (a); the converse follows trivially by applying (a).

■

R As illustracted above, convergence, continuity, (and compactness) can be defined by using open sets $\mathcal{T}$ only.

1.6.2. Topological Spaces

Definition 1.21 A **topological space** $(X, \mathcal{T})$ consists of a (non-empty) set X, and a family of subsets of X ("open sets" $\mathcal{T}$) such that

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T}$ implies $U \bigcap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$.

The elements in $\mathcal{T}$ are called **open subsets** of X. The $\mathcal{T}$ is called a **topology** on X.

■ **Example 1.20** 1. Let (X, d) be any metric space, and

$$\mathcal{T} = \{\text{all open subsets of } X\}$$

It's clear that $\mathcal{T}$ is a topology on X.

2. Define the discrete topology

$$\mathcal{T}_{\text{dis}} = \{\text{all subsets of } X\}$$

It's clear that $\mathcal{T}_{\text{dis}}$ is a topology on X, (which also comes from the discrete metric (X, d_{discrete})).

R We say $(X,\mathcal{T})$ is induced from a metric (X,d) (or it is **metrizable**) if $\mathcal{T}$ is the faimly of open subsets in (X,d).

3. Consider the indiscrete topology $(X,\mathcal{T}_{\text{indis}})$, where X contains more than one element:

$$\mathcal{T}_{\text{indis}} = \{\emptyset, X\}.$$

Question: is $(X,\mathcal{T}_{\text{indis}})$ metrizable? No. For any metric d defined on X, let x,y be distinct points in X, and then $\varepsilon := d(x,y) > 0$, hence $B_{\frac{1}{2}\varepsilon}(x)$ is a open set belonging to the corresponding induced topology. Since $x \in B_{\frac{1}{2}\varepsilon}(x)$ and $y \notin B_{\frac{1}{2}\varepsilon}(y)$, we conclude that $B_{\frac{1}{2}\varepsilon}(x)$ is neither $\emptyset$ nor X, i.e., the topology induced by any metric d is not the indiscrete topology.

4. Consider the cofinite topology $(X,\mathcal{T}_{\text{cofin}})$:

$$\mathcal{T}_{\text{cofin}} = \{U \mid X \setminus U \text{ is a finite set}\} \bigcup \{\emptyset\}$$

Question: is $(X,\mathcal{T}_{\text{cofin}})$ metrizable?

Definition 1.22 [Equivalence] Two metric spaces are **topologically equivalent** if they give rise to the same topology.

Example 1.21 Metrics d_1, d_2, d_∞ in $\mathbb{R}^n$ are topologically equivalent.

1.6.3. Closed Subsets

Definition 1.23 [Closed] Let $(X, \mathcal{T})$ be a topology space. Then $V \subseteq X$ is **closed** if $X \setminus V \in \mathcal{T}$

Example 1.22 Under the topology space $(\mathbb{R}, \mathcal{T}_{\text{usual}})$, $(b, \infty) \bigcup (-\infty, a) \in \mathcal{T}$. Therefore,

$$[a, b] = \mathbb{R} \setminus \left((b, \infty) \bigcup (-\infty, a) \right)$$

is closed in $\mathbb{R}$ under usual topology.

R It is important to say that V is **closed in** X**.** You need to specify the underlying the space X.

2.3. Monday for MAT4002

Reviewing.

1. Topological Space $(X, \mathcal{J})$: a special class of topological space is that induced from metric space (X,d):

$$(X, \mathcal{T}), \quad \text{with } \mathcal{T} = \{\text{all open sets in } (X,d)\}$$

2. Closed Sets $(X \setminus U)$ with U open.

Proposition 2.8 Let $(X,\mathcal{T})$ be a topological space,

1. $\emptyset, X$ are closed in X
2. V_1, V_2 closed in X implies that $V_1 \bigcup V_2$ closed in X
3. $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ closed in X implies that $\bigcap_{\alpha\in\mathcal{A}} V_\alpha$ closed in X

Proof. Applying the De Morgan's Law

$$(X \setminus \bigcup_{i\in I} U_i) = \bigcap_{i\in I}(X \setminus U_i)$$

∎

2.3.1. Convergence in topological space

Definition 2.4 [Convergence] A sequence $\{x_n\}$ of a topological space $(X,\mathcal{T})$ converges to $x \in X$ if $\forall U \ni x$ is open, there $\exists N$ such that $x_n \in U, \forall n \geq N$.

Example 2.9
1. The topology for the space $(X = \mathbb{R}^n, d_2) \to (X, \mathcal{T})$ (i.e., a topological space induced from meric space $(X = \mathbb{R}^n, d_2)$) is called a **usual topology** on $\mathbb{R}^n$. When I say $\mathbb{R}^n$ (or subset of $\mathbb{R}^n$) is a topological space, it is equipeed with usual topology.

   Convergence of sequence in $(\mathbb{R}^n, \mathcal{T})$ is the usual convergence in analysis.

   For $\mathbb{R}^n$ or metric space, the limit of sequence (if exists) is unique.

2. Consider the topological space $(X, \mathcal{T}_{\text{indiscrete}})$. Take any sequence $\{x_n\}$ in X, it is convergent to any $x \in X$. Indeed, for $\forall U \ni x$ open, $U = X$. Therefore,

$$x_n \in U(= X), \forall n \geq 1.$$

3. Consider the topological space $(X, \mathcal{T}_{\text{cofinite}})$, where X is infinite. Consider $\{x_n\}$ is a sequence satisfying $m \neq n$ implies $x_m \neq x_n$. Then $\{x_n\}$ is convergent to any $x \in X$. (Question: how to define openness for $\mathcal{T}_{\text{cofinite}}$ and $\mathcal{T}_{\text{indiscrete}}$)?
4. Consider the topological space $(X, \mathcal{T}_{\text{discrete}})$, the sequence $\{x_n\} \to x$ is equivalent to say $x_n = x$ for all sufficiently large n.

R The limit of sequences may not be unique. The reason is that "$\mathcal{T}$ is not big enough". We will give a criterion to make sure the limit is unique in the future. (Hausdorff)

Proposition 2.9 If $F \subseteq (X, \mathcal{T})$ is closed, then for any convergent sequence $\{x_n\}$ in F, the limit(s) are also in F.

Proof. Let $\{x_n\}$ be a sequence in F with limit $x \in X$. Suppose on the contrary that $x \notin F$ (i.e., $x \in X \setminus F$ that is open). There exists N such that

$$x_n \in X \setminus F, \forall n \geq N,$$

i.e., $x_n \notin F$, which is a contradiction. ∎

R The converse may not be true. If the $(X, \mathcal{T})$ is metrizable, the converse holds. Counter-example: Consider the co-countable topological space $(X = \mathbb{R}, \mathcal{T}_{\text{co-co}})$, where

$$\mathcal{T}_{\text{co-co}} = \{U \mid X \setminus U \text{ is a countable set}\} \bigcup \{\emptyset\},$$

and X is uncontable. Then note that $F = [0,1] \subsetneq X$ is an un-countable set, and under co-countable topology, $F \supseteq \{x_n\} \to x$ implies $x_n = x \in F$ for all n. It's clear that $X \setminus F \notin \mathcal{T}_{\text{co-co}}$, i.e., F is not closed.

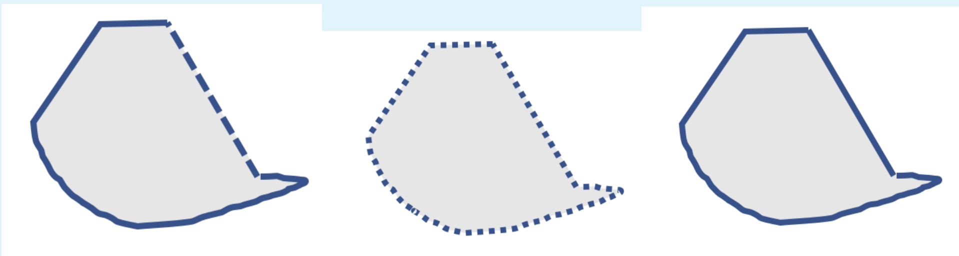
2.3.2. Interior, Closure, Boundary

Definition 2.5 Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.

1. The **interior** of A is
$$A^\circ = \bigcup_{U \subseteq A, U \text{ is open}} U$$
2. The **closure** of A is
$$\overline{A} = \bigcap_{A \subseteq V, V \text{ is closed}} V$$

If $\overline{A} = X$, we say that A is dense in X.

The graph illustration of the definition above is as follows:



(a) Illustration of A (b) Illustration of A° (c) Illustration of $\overline{A}$

Figure 2.1: Graph Illustrations

■ **Example 2.10**

1. For $[a,b) \subseteq \mathbb{R}$, we have:
$$[a,b)^\circ = (a,b), \quad \overline{[a,b)} = [a,b]$$
2. For $X = \mathbb{R}$, $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$.
3. Consider the discrete topology $(X, \mathcal{T}_{\text{discrete}})$, we have
$$S^\circ = S, \quad \overline{S} = S$$

The insights behind the definition (2.5) is as follows

Proposition 2.10 1. A° is the largest open subset of X contained in A;

$\overline{A}$ is the smallest closed subset of X containing A.

2. If $A \subseteq B$, then $A^\circ \subseteq B$ and $\overline{A} \subseteq \overline{B}$
3. A is open in X is equivalent to say $A^\circ = A$; A is closed in X is equivalent to say $\overline{A} = A$.

■ **Example 2.11** Let (X,d) be a metric space. What's the closure of an open ball $B_r(x)$?

The direct intuition is to define the closed ball

$$\bar{B}_r(x) = \{y \in X \mid d(x,y) \le r\}.$$

Question: is $\bar{B}_r(x) = \overline{B_r(x)}$?

1. Since $\bar{B}_r(x)$ is a closed subset of X, and $B_r(x) \subseteq \bar{B}_r(x)$, we imply that

$$\overline{B_r(x)} \subseteq \bar{B}_r(x)$$

2. Howover, we may find an example such that $\overline{B_r(x)}$ is a proper subset of $\bar{B}_r(x)$: Consider the discrete metric space (X, d_{discrete}) and for $\forall x \in X$,

$$B_1(x) = \{x\} \implies \overline{B_1(x)} = \{x\}, \quad \bar{B}_1(x) = X$$

The equality $\bar{B}_r(x) = \overline{B_r(x)}$ holds when (X,d) is a normed space.

■

Here is another characterization of $\overline{A}$:

Proposition 2.11

$$\overline{A} = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \emptyset\}$$

Proof. Define

$$S = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \emptyset\}$$

It suffices to show that $\overline{A} = S$.

1. First show that S is closed:

$$X \setminus S = \{x \in X \mid \exists U_x \ni x \text{ open s.t. } U_x \bigcap A = \emptyset\}$$

Take $x \in X \setminus S$, we imply there exists open $U_x \ni x$ such that $U_x \bigcap A = \emptyset$. We claim $U_x \subseteq X \setminus S$:

   - For $\forall y \in U_x$, note that $U_x \ni y$ that is open, such that $U_x \bigcap A = \emptyset$. Therefore, $y \in X \setminus S$.

   Therefore, we have $x \in U_x \subseteq X \setminus S$ for any $\forall x \in X \setminus S$.

   Note that

$$X \setminus S = \bigcup_{x \in X \setminus S} \{x\} \subseteq \bigcup_{x \in X \setminus S} U_x \subseteq X \setminus S,$$

   which implies $X \setminus S = \bigcup_{x \in X \setminus S} U_x$ is open, i.e., S is closed in X.

2. By definition, it is clear that $A \subseteq S$:

$$\forall a \in A, \forall \text{open } U \ni a, U \bigcap A \supseteq \{a\} \neq \emptyset \implies a \in S.$$

   Therefore, $\overline{A} \subseteq \overline{S} = S$.

3. Suppose on the contrary that there exists $y \in S \setminus \overline{A}$.

   Since $y \notin \overline{A}$, by definition, there exists $F \supseteq A$ closed such that $y \notin F$.

   Therefore, $y \in X \setminus F$ that is open, and

$$(X \setminus F) \bigcap A \subseteq (X \setminus A) \bigcap A = \emptyset \implies y \notin S,$$

   which is a contradiction. Therefore, $S = \overline{A}$.

∎

Definition 2.6 [accumulation point] Let $A \subseteq X$ be a subset in a topological space. We call $x \in X$ are an **accumulation point (limit point)** of A if

$$\forall U \subseteq X \text{ open s.t. } U \ni x, (U \setminus \{x\}) \bigcap A \neq \emptyset.$$

The set of accumulation points of A is denoted as A'

Proposition 2.12 $\overline{A} = A \bigcup A'$.

Proof. This proposition directly follows from Proposition (2.11) and the definition of A′. ■

2.6. Wednesday for MAT4002

Reviewing.

1. Interior, Closure:

$$\overline{A} = \{x \mid \forall U \ni x \text{ open}, U \bigcap A \neq \emptyset\}$$

2. Accumulation points

2.6.1. Remark on Closure

Definition 2.14 [Sequential Closure] Let A_S be the set of limits of any convergent sequence in A, then A_S is called the **sequential closure** of A.

Definition 2.15 [Accumulation/Cluster Points] The set of accumulation (limit) points is defined as

$$A' = \{x \mid \forall U \ni x \text{ open }, (U \setminus \{x\}) \bigcap A \neq \emptyset\}$$

R

1. (a) There exists some point in A but not in A':

$$A = \{1, 2, 3, \dots, n, \dots\}$$

Then any point in A is not in A'

(b) There also exists some point in A' but not in A:

$$A = \{\frac{1}{n} \mid n \geq 1\}$$

Then the point 0 is in A' but not in A.

2. The closure $\overline{A} = A \bigcup A'$.
3. The size of the sequencial closure A_S is between A and $\overline{A}$, i.e., $A \subseteq A_S \subseteq \overline{A}$:

It's clear that $A \subseteq A_S$, since the sequence $\{a_n := a\}$ is convergent to a for

$\forall a \in A$.

For all $a \in A_S$, we have $\{a_n\} \to a$. Then for any open $U \ni a$, there exists N such that $\{a_N, a_{N+1}, \dots\} \subseteq U \bigcap A \neq \emptyset$. Therefore, $a \in \overline{A}$, i.e., $A_S \subseteq \overline{A}$.

Question: Is $A_S = \overline{A}$?

Proposition 2.21 Let (X,d) be a metric space, then $A_S = \bar{A}$.

Proof. Let $a \in \overline{A}$, then there exists $a_n \in B_{1/n}(a) \bigcap A$, which implies $\{a_n\} \to a$, i.e., $a \in A_S$. ∎

R If $(X, \mathcal{T})$ is metrizable, then $A_S = \overline{A}$. The same goes for first countable topological spaces. However, A_S is a proper subset of $\overline{A}$ in general:

Let $A \subseteq X$ be the set of continuous functions, where $X = \mathbb{R}^{\mathbb{R}}$ denotes the set of all real-valued functions on $\mathbb{R}$, with the topology of pointwise convergence.

Then $A_S = B_1$, the set of all functions of first Baire-Category on $\mathbb{R}$; and $[A_S]_S = B_2$, the set of all functions of second Baire-Category on $\mathbb{R}$. Since $B_1 \neq B_2$, we have $[A_S]_S = A_S$. Note that $\overline{\overline{A}} = \overline{A}$. We conclude that A_S cannot equal to $\overline{A}$, since the sequential closure operator cannot be idemotenet.

Definition 2.16 [Boundary] The **boundary** of $\boldsymbol{A}$ is defined as

$$\partial \boldsymbol{A} = \overline{A} \setminus A^\circ$$

Proposition 2.22 Let $(X, \mathcal{T})$ be a topological space with $A, B \subseteq X$.

$$\overline{X \setminus A} = X \setminus A^\circ, \quad (X \setminus B)^\circ = X \setminus \overline{B} \quad \partial A = \overline{A} \bigcap (\overline{X \setminus A})$$

Proof.

$$X \setminus A^\circ = X \setminus \left(\bigcup_{U \text{ is open, } U \subseteq A} U \right) \tag{2.2a}$$

$$= \bigcap_{U \text{ is open, } U \subseteq A} (X \setminus U) \tag{2.2b}$$

$$= \bigcap_{V \text{ is closed, } F \supseteq X \setminus A} F \tag{2.2c}$$

$$= \overline{X \setminus A} \tag{2.2d}$$

Denoting $X \setminus A$ by B, we obtain:

$$(X \setminus B)^\circ = A^\circ \tag{2.3a}$$

$$= X \setminus (X \setminus A^\circ) \tag{2.3b}$$

$$= X \setminus \overline{X \setminus A} \tag{2.3c}$$

$$= X \setminus \overline{B} \tag{2.3d}$$

By definition of ∂A,

$$\partial A = \overline{A} \setminus A^\circ \tag{2.4a}$$

$$= \overline{A} \bigcap (X \setminus A^\circ) \tag{2.4b}$$

$$= \overline{A} \bigcap (\overline{X \setminus A}) \tag{2.4c}$$

■

2.6.2. Functions on Topological Space

Definition 2.17 [Continuous] Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map. Then the function f is continuous, if

$$U \in \mathcal{T}_Y \implies f^{-1}(U) \in \mathcal{T}_X$$

■ **Example 2.16**

1. The identity map $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined as $x \mapsto x$ is continuous
2. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{discrete}}) \to (X, \mathcal{T}_{\text{indiscrete}})$ defined as $x \mapsto x$ is continuous. Since $\mathrm{id}^{-1}(\emptyset) = \emptyset$ and $\mathrm{id}^{-1}(X) = X$
3. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{indiscrete}}) \to (X, \mathcal{T}_{\text{discrete}})$ defined as $x \mapsto x$ is not continuous.

Proposition 2.23 If $f : X \to Y$, and $g : Y \to Z$ be continuous, then $g \circ f$ is continuous

Proof. For given $U \in \mathcal{T}_Z$, we imply

$$g^{-1}(U) \in \mathcal{T}_Y \implies f^{-1}(g^{-1}(U)) \in \mathcal{T}_X,$$

i.e., $(g \circ f)^{-1}(U) \in \mathcal{T}_X$ ∎

Proposition 2.24 Suppose $f : X \to Y$ is continuous between two topological spaces. Then $\{x_n\} \to x$ implies $\{f(x_n)\} \to f(x)$.

Proof. Take open $U \ni f(x)$, which implies $f^{-1}(U) \ni x$. Since $f^{-1}(U)$ is open, we imply there exists N such that

$$\{x_n \mid n \geq N\} \subseteq f^{-1}(U),$$

i.e., $\{f(x_n) \mid n \geq N\} \subseteq U$ ∎

We use the notion of Homeomorphism to describe the equivalence between two topological spaces.

Definition 2.18 [Homeomorphism] A **homeomorphism** between spaces topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a bijection

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y),$$

such that both f and f^{-1} are continuous.

2.6.3. Subspace Topology

Definition 2.19 Let $A \subseteq X$ be a non-empty set. The **subspace topology** of A is defined as:

1. $\mathcal{T}_A := \{U \bigcap A \mid U \in \mathcal{T}_A\}$
2. The **coarsest** topology on A such that the **inclusion map**

$$i : (A, \mathcal{T}_A) \to (X, \mathcal{T}_X), \quad i(x) = x$$

   is continuous.

   (We say the topology $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, or $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ e.g., $\mathcal{T}_{\text{discrete}}$ is the finest topology, and $\mathcal{T}_{\text{indiscrete}}$ is coarsest topology.)

3. The (**unique**) topology such that for any $(Y, \mathcal{T}_Y)$,

$$f : (Y, \mathcal{T}_Y) \to (A, \mathcal{T}_A)$$

   is continuous iff $i \circ f : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ (where i is the inclusion map) is continuous.

Proposition 2.25 The definition (1) and (2) in (2.19) are equivalent.

Outline. The proof is by applying

$$i^{-1}(S) = S \bigcap A, \quad \forall S$$

∎

Example 2.17 Let all English and numerical letters be subset of $\mathbb{R}^2$:

P, 6

The homeomorphism can be constrcuted between these two English letters.

Proposition 2.26 The definition (2) and (3) in (2.19) are equivalent.

Proof. Necessity.

- For $\forall U \in \mathcal{T}_X$, consider that

$$(i \circ f)^{-1}(U) = f^{-1}(i^{-1}(U)) = f^{-1}(U \bigcap A)$$

since $U \bigcap A \in \mathcal{T}_A$ and f is continuous, we imply $(i \circ f)^{-1}(U) \in \mathcal{T}_Y$

- For $\forall U' \in \mathcal{T}_A$, we have $U' = U \bigcap A$ for some $U \in \mathcal{T}_X$. Therefore,

$$f^{-1}(U') = f^{-1}(U \bigcap A) = f^{-1}(i^{-1}(U)) = (i \circ f)^{-1}(U) \in \mathcal{T}_Y.$$

The sufficiency is left as exercise. ■

Proposition 2.27

1. The definition (1) in (2.19) does define a topology of A
2. Closed sets of A under subspace topology are of the form $V \bigcap A$, where V is closed in X

Proposition 2.28 Suppose $(A, \mathcal{T}_A) \subseteq (X, \mathcal{T}_X)$ is a subspace topology, and $B \subseteq A \subseteq X$. Then

1. $\bar{B}^A = \bar{B}^X \bigcap A$.
2. $B^{\circ A} \supseteq B^{\circ X}$

Proof. By proposition (2.27), $\bar{B}^X \bigcap A$ is closed in A, and $\bar{B}^X \bigcap A \supset B$, which implies

$$\bar{B}^A \subseteq \bar{B}^X \bigcap A$$

Note that $\bar{B}^A \supset B$ is closed in A, which implies $\bar{B}^A = V \bigcap A \subseteq V$, where V is closed in X. Therefore,

$$\bar{B}^X \subseteq V \implies \bar{B}^X \bigcap A \subseteq V \bigcap A = \bar{B}^A$$

Therefore, $\bar{B}^A = \bar{B}^X \subseteq V$

■

Can we have $B^{\circ X} = B^{\circ A}$?

2.6.4. Basis (Base) of a topology

Roughly speaking, a basis of a topology is a family of "generators" of the topology.

Definition 2.20 Let $(X, \mathcal{T})$ be a topological space. A family of subsets $\mathcal{B}$ in X is a **basis** for $\mathcal{T}$ if

1. $\mathcal{B} \subseteq \mathcal{T}$, i.e., everything in $\mathcal{B}$ is open
2. Every $U \in \mathcal{T}$ can be written as union of elements in $\mathcal{B}$.

■

■ **Example 2.18** 1. $\mathcal{B} = \mathcal{T}$ is a basis.

2. For $X = \mathbb{R}^n$,

$$\mathcal{B} = \{B_r(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{Q}^n, r \in \mathbb{Q} \bigcap (0, \infty)\}$$

   Exercise: every $(a, b) = \bigcup_{i \in I}(p_i, q_i)$ for $p_i, q_i \in \mathbb{Q}$.

   Therefore, $\mathcal{B}$ is countable.

■

Proposition 2.29 If $(X, \mathcal{T})$ has a countable basis, e.g., $\mathbb{R}^n$, then $(X, \mathcal{T})$ has a second-countable space.

3.3. Monday for MAT4002

3.3.1. Remarks on Basis and Homeomorphism

Reviewing.

1. $A \subseteq A_S \subseteq \overline{A}$, where A_S is sequential closure and $\overline{A}$ denotes closure.
2. Subspace topology.
3. Homeomorphism. Consider the mapping $f : X \to Y$ with the topogical space X, Y shown below, with the standard topology, the question is whether f is continuous?

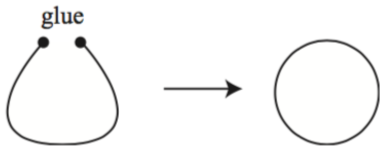


Figure 3.1: Diagram for mapping f

The answer is no, since the left in (3.1) can be isomorphically mapped into $(0,1)$; the right can be isomorphically mapped into $[0,1]$, and the mapping $(0,1) \to [0,1]$ cannot be isomorphism:

Proof. Assume otherwise the mapping $g : (0,1) \to [0,1]$ is isomorphism, and therefore $f^{-1}(U)$ is open for any open set U in the space $[0,1]$.
Construct $U = (1-\delta, 1]$ for $\delta \le 1$, and therfore $f^{-1}((1-\delta,1])$ is open, and therfore for the point $x = f^{-1}(1)$, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq f^{-1}((1-\delta,1]) \implies [x-\varepsilon, x) \subseteq f^{-1}((1-\delta,1)), \text{ and } (x, x+\varepsilon] \subseteq f^{-1}((1-\delta,1)).$$

which implies that there exists a, b such that $[x-\varepsilon, x) = f^{-1}((a,1))$ and $(x, x+\varepsilon] = f^{-1}((b,1))$, i.e., $f^{-1}((a,b) \cap (b,1))$ admits into two values in $[x-\varepsilon, x)$ and $(x, x+\varepsilon]$, which is a contradiction. ■

4. Basis of a topology $\mathcal{B} \subseteq (X, \mathcal{T})$ is a collection of open sets in the space such that the whole space can be recovered, or equivalently

(a) $\mathcal{B} \subseteq \mathcal{T}$

(b) Every set in $\mathcal{T}$ can expressed as a union of sets in $\mathcal{B}$

Example: Let $\mathbb{R}^n$ be equipped with usual topology, then

$$B = \{B_q(x) \mid x \in \mathbb{Q}^n, q \in \mathbb{Q}^+\} \text{ is a basis of } \mathbb{R}^n.$$

It suffices to show $U \subseteq \mathbb{R}^n$ can be written as

$$U = U_{x \in \mathbb{Q}} B_{q_x}(x)$$

Proposition 3.4 Let X, Y be topological spaces, and $\mathcal{B}$ a basis for topology on Y. Then

$$f : X \to Y \text{ is continuous} \iff f^{-1}(B) \text{ is open in X, } \forall B \in \mathcal{B}$$

Therefore checking $f^{-1}(U)$ is open for all $U \in \mathcal{T}_Y$ suffices to checking $f^{-1}(N)$ is open for all $B \in \mathcal{B}$.

Proof. The forward direction follows from the fact $B \subseteq \mathcal{T}_Y$.

To show the reverse direction, let $U \in \mathcal{T}_Y$, then $U = \bigcup_{i \in I} B_i$, where $B_i \in \mathcal{B}$, which implies

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

which is open in X by our hypothesis. ■

Corollary 3.1 Let $f : X \to Y$ be a bijection. Suppose there is a basis $\mathcal{B}_X$ of $\mathcal{T}_X$ such that $\{f(B) \mid B \in \mathcal{B}_X\}$ forms a basis of $\mathcal{T}_Y$. Then $X \cong Y$.

Proof. Suppose $W \in \mathcal{T}_Y$, then by our hypothesis,

$$W = \bigcup_{i \in I} f(B_i),\ B_i \in \mathcal{B}_X \implies f^{-1}(W) = \bigcup_{i \in I} B_i \in \mathcal{T}_X,$$

which implies f is continuous.

Suppose $U \in \mathcal{T}_X$, then

$$U = \bigcup_{i \in I} B_i \implies f(U) = \bigcup_{i \in I} f(B_i) \in \mathcal{T}_Y \implies [f^{-1}]^{-1}(U) \in \mathcal{T}_Y,$$

i.e., f^{-1} is continuous. ∎

Question: *how to recognise whether a family of subsets is a basis for some given topology?*

Proposition 3.5 Let X be a set, $\mathcal{B}$ is a collection of subsets satisfying

1. X is a union of sets in $\mathcal{B}$, i.e., every $x \in X$ lies in some $B_x \in \mathcal{B}$
2. The intersection $B_1 \cap B_2$ for $\forall B_1, B_2 \in \mathcal{B}$ is a union of sets in $\mathcal{B}$, i.e., for each $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the collection of subsets $\mathcal{T}_{\mathcal{B}}$, formed by taking any union of sets in $\mathcal{B}$, is a topology, and $\mathcal{B}$ is a basis for $\mathcal{T}_B$.

Proof. 1. $\emptyset \in \mathcal{T}_{\mathcal{B}}$ (taking nothing from $\mathcal{B}$); for $x \in X, B_x \in \mathcal{B}$, by hypothesis (1),

$$X = \bigcup_{x \in X} B_x \in \mathcal{T}_{\mathcal{B}}$$

2. Suppose $T_1, T_2 \in \mathcal{T}_{\mathcal{B}}$. Let $x \in T_1 \cap T_2$, where T_i is a union of subsets in $\mathcal{B}$. Therefore,

$$\begin{cases} x \in B_1 \subseteq T_1, & B_1 \in \mathcal{B} \\ x \in B_2 \subseteq T_2, & B_2 \in \mathcal{B} \end{cases}$$

which implies $x \in B_1 \cap B_2$, i.e., $x \in B_x \subseteq B_1 \cap B_2$ for some $B_x \in \mathcal{B}$. Therefore,

$$\bigcup_{x \in B_1 \cap B_2} \{x\} \subseteq \bigcup_{x \in B_1 \cap B_2} B_x \subseteq B_1 \cap B_2,$$

i.e., $B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_x$, i.e., $B_1 \cap B_2 \in \mathcal{T}_{\mathcal{B}}$.

3. The property that $\mathcal{T}_{\mathcal{B}}$ is closed under union operations can be checked directly.

The proof is complete. ∎

3.3.2. Product Space

Now we discuss how to construct new topological spaces out of given ones is by taking Cartesian products:

Definition 3.4 Let $(X,\mathcal{T}_X),(Y,\mathcal{T}_Y)$ be topological spaces. Consider the family of subsets in $X\times Y$:

$$\mathcal{B}_{X\times Y}=\{U\times V\mid U\in\mathcal{T}_X,V\in\mathcal{T}_y\}$$

This $\mathcal{B}_{X\times Y}$ forms a basis of a topology on $X\times Y$. The induced topology from $\mathcal{B}_{X\times Y}$ is called **product topology**.

For example, for $X=\mathbb{R},Y=\mathbb{R}$, the elements in $\mathcal{B}_{X\times Y}$ are rectangles.

Proof for well-definedness in definition (3.4). We apply proposition (3.5) to check whether $B_{X\times Y}$ forms a basis:

1. For any $(x,y)\in X\times Y$, we imply $x\in X,y\in Y$. Note that $X\in\mathcal{T}_X,Y\in\mathcal{T}_Y$, we imply $(x,y)\in X\times Y\in\mathcal{B}_{X\times Y}$.
2. Suppose $U_1\times V_1,U_2\times V_2\in\mathcal{B}_{X\times Y}$, then

$$(U_1\times V_1)\cap(U_2\times V_2)=(U_1\cap U_2)\times(V_1\cap V_2),$$

where $U_1\cap U_2\in\mathcal{T}_X,V_1\cap V_2\in\mathcal{T}_Y$. Therefore, $(U_1\times V_1)\cap(U_2\times V_2)\in\mathcal{B}_{X\times Y}$.

∎

R However, the product topology may not necessarily become the largest topology in the space $X\times Y$. Consider $X=\mathbb{R},Y=\mathbb{R}$, the open set in the space $X\times Y$ may not necessarily be rectangles. However, all elements in $\mathcal{B}_{X\times Y}$ are rectangles.

Example 3.8 The space $\mathbb{R}\times\mathbb{R}$ is homeomorphic to $\mathbb{R}^2$, where the product topology is defined on $\mathbb{R}\times\mathbb{R}$ and the standard topology is defined on $\mathbb{R}^2$:

Construct the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ with $(a,b)\to(a,b)$.

Obviously, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ is a bijection.

Take the basis of the topology on $\mathbb{R}$ as open intervals,

$$B_X = \{(a,b) \mid a < b \text{ in } \mathbb{R}\}$$

Therefore, one can verify that the set $\mathcal{B} := \{(a,b) \times (c,d) \mid a < b, c < d\}$ forms a basis for the product topology, and

$$\{f(B) \mid B \in \mathcal{B}\} = \{(a,b) \times (c,d) \mid a < b, c < d\}$$

forms a basis of the usual topology in $\mathbb{R}^2$.

By Corollary (3.1), we imply $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$. ■

We also raise an example on the homeomorphism related to product spaces:

■ **Example 3.9** Let $S^1 = \{(\cos x, \sin x \mid x \in [0, 2\pi])\}$ be a unit circle on $\mathbb{R}^2$.

Consider $f : S^1 \times (0, \infty) \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ defined as

$$f(\cos x, \sin x, r) \mapsto (r \cos x, r \sin x)$$

It's clear that f is a bijection, and f is continuous. Moreover, the inverse $g := f^{-1}$ is defined as

$$g(a,b) = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, \sqrt{a^2 + b^2}\right)$$

which is continuous as well. Therefore, the $f : S^1 \times (0, \infty) \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is a homeomorphism.

■

3.6. Wednesday for MAT4002

3.6.1. Remarks on product space

Reviewing.

- Product Topology: For topological space $(X, \mathcal{T}_X)$ and $(Y, \mathcal{Y})$, define the basis

$$\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

  and the family of union of subsets in $\mathcal{B}_{X\times Y}$ forms a product topology.

Proposition 3.9 a ring torus is homeomorphic to the Cartesian product of two circles, say $S^1 \times S^1 \cong T$.

Proof. Define a mapping $f : [0,2\pi]\times[0,2\pi] \to T$ as

$$f(\theta,\phi) = \Big((R + r\cos\theta)\cos\phi, \quad (R + r\cos\theta)\sin\phi, \quad r\sin\theta\Big)$$

Define $i : T \to \mathbb{R}^3$, we imply

$$i \circ f : [0,2\pi]\times[0,2\pi] \to \mathbb{R}^3 \text{ is continuous}$$

Therefore we imply $f : [0,2\pi]\times[0,2\pi] \to T$ is continuous. Together with the condition that

$$\begin{cases} f(0,y) = f(2\pi,y) \\ f(x,0) = f(x,2\pi) \end{cases}$$

we imply the function $f : S^1 \times S^1 \to T$ is continuous. We can also show it is bijective. We can also show f^{-1} is continuous. ■

Proposition 3.10 1. Let $X \times Y$ be endowed with product topology. The projection mappings defined as

$$p_X : X \times Y \to X, \text{ with } p_X(x,y) = x$$

$$p_Y : X \times Y \to Y, \text{ with } p_Y(x,y) = y$$

are continuous.

2. (an equivalent definition for product topology) The product topology is the **coarest topology** on $X \times Y$ such that p_X and p_Y are both continuous.
3. (an equivalent definition for product topology) Let Z be a topological space, then the product topology is the unique topology that the red and the blue line in the diagram commutes:

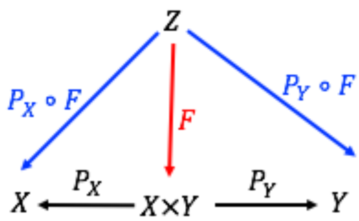


Figure 3.3: Diagram summarizing the statement (*)

namely,

the mapping $F : Z \to X \times Y$ is continuous iff both $P_X \circ F : Z \to X$ and $P_Y \circ F : Z \to Y$ are continuous. (*)

Proof. 1. For any open U, we imply $p_X^{-1}(U) = U \times Y \in \mathcal{B}_{X\times Y} \subseteq \mathcal{T}_{X\times Y}$, i.e., $p_X^{-1}(U)$ is open. The same goes for p_Y.

2. It suffices to show any topology $\mathcal{T}$ that meets the condition in (2) must contain $\mathcal{T}_{\text{product}}$. We imply that for $\forall U \in \mathcal{T}_X, V \in \mathcal{T}_Y$,

$$\begin{cases} p_X^{-1}(U) = U \times X \in \mathcal{T} \\ p_Y^{-1}(V) = X \times V \in \mathcal{T} \end{cases} \implies (U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V \in \mathcal{T},$$

which implies $\mathcal{B}_{X\times Y} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is closed for union operation on subsets, we imply $\mathcal{T}_{\text{product topology}} \subseteq \mathcal{T}$.

3. (a) Firstly show that $\mathcal{T}_{\text{product}}$ satisfies (*).

   - For the forward direction, by (1) we imply both $p_X \circ F$ and $p_Y \circ F$ are continuous, since the composition of continuous functions are continuous as well.
   - For the reverse direction, for $\forall U \times \mathcal{T}_X, V \in \mathcal{T}_Y$,

   $$F^{-1}(U \times V) = (p_X \circ F)^{-1}(X) \cap (p_Y \circ F)^{-1}(Y),$$

   which is open due to the continuity of $p_X \circ F$ and $p_Y \circ F$.

   (b) Then we show the uniqueness of $\mathcal{T}_{\text{product}}$. Let $\mathcal{T}$ be another topology $X \times Y$ satisfying (*).

   - Take $Z = (X \times Y, \mathcal{T})$, and consider the identity mapping $F = \text{id} : Z \to Z$, which is continuous. Therefore $p_X \circ \text{id}$ and $p_Y \circ \text{id}$ are continuous, i.e., p_X and p_Y are continuous. By (2) we imply $\mathcal{T}_{\text{product}} \subseteq \mathcal{T}$.
   - Take $Z = (X \times Y, \mathcal{T}_{\text{product}})$, and consider the identity mapping $F = \text{id} : Z \to Z$. Note that $p_X \circ F = p_X$ and $p_Y \circ F = p_Y$, which is continuous by (1). Therefore, the identity mapping $F : (X \times Y, \mathcal{T}_{\text{product}}) \to (X \times Y, \mathcal{T})$ is continuous, which implies

   $$U = \text{id}^{-1}(U) \subseteq \mathcal{T}_{\text{product}} \text{ for } \forall U \in \mathcal{T},$$

   i.e., $\mathcal{T} \subseteq \mathcal{T}_{\text{product}}$.

   The proof is complete.

■

Definition 3.6 [Disjoint Union] Let $X \times Y$ be two topological spaces, then the **disjoint union** of X and Y is

$$X \coprod Y := (X \times \{0\}) \cup (Y \times \{1\})$$

R

1. We define that U is open in $X \coprod Y$ if

   (a) $U \cap (X \times \{0\})$ is open in $X \times \{0\}$; and

   (b) $U \cap (Y \times \{1\})$ is open in $Y \times \{1\}$.

   We also need to show the well-definedness for this definition.

2. S is open in $X \coprod Y$ iff S can be expressed as

$$S = (U \times \{0\}) \cup (V \times \{1\})$$

   where $U \subseteq X$ is open and $V \subseteq Y$ is open.

3.6.2. Properties of Topological Spaces

3.6.2.1. Hausdorff Property

Definition 3.7 [First Separation Axiom] A topological space X satisfies the **first separation axiom** if for any two distinct points $x \neq y \in X$, there exists open $U \ni x$ but not including y.

Proposition 3.11 A topological space X has first separation property if and only if for $\forall x \in X$, $\{x\}$ is closed in X.

Proof. *Sufficiency.* Suppose that $x \neq y$, then construct $U := X \setminus \{y\}$, which is a open set that contains x but not includes y.

Necessity. Take any $x \in X$, then for $\forall y \neq x$, there exists $y \in U_y$ that is open and $x \notin U_y$. Thus

$$\{y\} \subseteq U_y \subseteq X \setminus \{x\}$$

which implies

$$\bigcup_{y \in X \setminus \{x\}} \{y\} \subseteq \bigcup_{y \in X \setminus \{x\}} U_y \subseteq X \setminus \{x\},$$

i.e., $X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} U_y$ is open in X, i.e., $\{x\}$ is closed in X. ∎

Definition 3.8 [Second separation Axiom] A topological space satisfies the **second separation axiom** (or X is Hausdorff) if for all $x \neq y$ in X, there exists open sets U,V such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset$$

■ **Example 3.13** All metrizable topological spaces are Hausdorff.

Suppose $d(x,y) = r > 0$, then take $B_{r/2}(x)$ and $B_{r/2}(y)$ ■

■ **Example 3.14** Note that a topological space that is **first separable** may not necessarily be **second separable**:

Consider $\mathcal{T}_{\text{co-finite}}$, then X is first separable but not Hausdorff:

Suppose on the contrary that for given $x \neq y$, there exists open sets U,V such that $x \in U, y \in V$, and

$$U \cap V = \emptyset \implies X = X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V),$$

implying that the union of two finite sets equals X, which is infinite, which is a contradiction.

■

4.3. Monday for MAT4002

There will be a quiz next Monday. The scope is everything before CNY holiday. There will be one question with four parts for 40 minutes.

4.3.1. Hausdorffness

Reviewing. A topological space $(X,\mathcal{T})$ is said to be **Hausdorff** (or satisfy the second separtion property), if given any distinct points $x,y\in X$, there exist disjoint open sets U,V such that $U\ni x$ and $V\ni y$.

Proposition 4.5 If the topological space $(X,\mathcal{T})$ is Hausdorff, then all sequences $\{x_n\}$ in X has at most one limit.

Proof. Suppose on the contrary that

$$\{x_n\}\to a,\quad \{x_n\}\to b,\text{ with }a\neq b$$

By separation property, there exists $U,V\in\mathcal{T}$ and $U\cap V=\emptyset$ such that $U\ni a$ and $V\ni b$.

By tje openness of U, there exists N such that $\{x_N,x_{N+1},\dots\}\subseteq U$, since $\{x_n\}\to a\in U$. Similarly, there exists M such that $\{x_M,x_{M+1},\dots\}\subseteq V$. Take $K=\max\{M,N\}+1$, then $\emptyset\neq U\cap V\ni x_K$, which is a contradiction. ∎

Proposition 4.6 Let X,Y be Hausdorff spaces. Then $X\times Y$ is Hausdorff with product topology.

Proof. Suppose that $(x_1,y_1)\neq(x_2,y_2)$ in $X\times Y$. Then $x_1\neq x_2$ or $y_1\neq y_2$. w.l.o.g., assume that $x_1\neq x_2$, then there exists U,V open in X such that $x_1\in U,x_2\in V$ with $U\cap V=\emptyset$.

Therefore, we imply $(U\times Y),(V\times Y)\in\mathcal{T}_{X\times Y}$, and

$$(U\times Y)\cap(V\times Y)=(U\cap V)\cap Y=\emptyset$$

with $(x_1,y_1)\in U\times Y,(x_2,y_2)\in V\times Y$, i.e., $X\times Y$ is Hausdorff with product topology. ∎

The same argument applies if the second separation property is replaced by first separation property.

Proposition 4.7 If $f : X \to Y$ is an injective continuous mapping, then Y is Hausdorff implies X is Hausdorff.

Proof. Suppose that Y satisfies the second separation property. For given $a \neq b$ in X, we imply $f(a) \neq f(b)$ in Y. Therefore, there exists $U \ni f(a), V \ni f(b)$ with $U \cap V = \emptyset$. It follows that

$$a \in f^{-1}(U), b \in f^{-1}(V), \quad f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset,$$

i.e., X is Hausdorff. ■

Corollary 4.1 If $f : X \to Y$ is homeomorphic, then X is Hausdorff iff Y is Hausdorff, i.e., Hausdorffness is a topological property (i.e., a property that is preserved under homeomorphism).

4.3.2. Connectedness

Definition 4.4 [Connected] The topological space $(X, \mathcal{T})$ is **disconnected** if there are open $U, V \in \mathcal{T}$ such that

$$U \neq \emptyset, V \neq \emptyset, \quad U \cap V = \emptyset, \quad U \cup V = X. \tag{4.4}$$

If no such $U, V \in \mathcal{T}$ exist, then X is **connected**.

Proposition 4.8 Let $(X, \mathcal{T})$ be topological spaces. TFAE (i.e., the followings are equivalent):

1. X is connected
2. The **only** subset of X which are both open and closed are $\emptyset$ and X
3. Any continuous function $f : X \to \{0,1\}$ ($\{0,1\}$ is equipped with discrete topology) is a constant function.

Proof. (1) implies (2): Suppose that $U \subseteq X$ is both open and closed. Then $U, X \setminus U$ are both open and disjoint, and $U \cup (X \setminus U) = X$. By connnectedness, either $U = \emptyset$ or $X \setminus U = \emptyset$. Therefore, $U = \emptyset$ or X.

(2) implies (3): Note that $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are open disjoint sets in X satisfying $U \cup V = X$. By the connectedness of X, either $(U,V) = (X,\emptyset)$ or $(V,U) = (\emptyset, X)$. In either case, we imply f is a constant function.

(3) implies (2): Suppose that $U \subseteq X$ is both open and closed. Construct the mapping

$$f(x) = \begin{cases} 0, & x \in U \\ 1, & x \in X \setminus U \end{cases}$$

It's clear that f is continuous, and therefore $f(x) = 0$ or 1. Therefore $U = \emptyset$ or X.

(2) implies (1): Suppose on the contrary that there exists open U, V such that (4.4) holds. By (4.4), we imply $U = X \setminus V$ is closed as well. Since $U \neq \emptyset$ and $U = \emptyset$ or X, we imply $U = X$, which implies $V = \emptyset$, which is a contradiction. ■

Corollary 4.2 The interval $[a,b] \subseteq \mathbb{R}$ is connnected

Proof. Suppose on the contrary that there exists continuous function $f : [a,b] \to \{0,1\}$ that takes 2 values. Construct the mapping $\tilde{f} : [a,b] \to \mathbb{R}$

$$\begin{aligned} &\tilde{f} : [a,b] \xrightarrow{f} \{0,1\} \xrightarrow{i} \mathbb{R}, \\ \text{with} \quad &\tilde{f} = i \circ f. \end{aligned}$$

Note that $\{0,1\} \subseteq \mathbb{R}$ denotes the subspace topology, we imply the inclusion mapping $i : \{0,1\} \to \mathbb{R}$ with $s \mapsto s$ is continuous. The composition of continuous mappings is continuous as well, i.e., $\tilde{f}$ is continuous.

Since the function f can take two values, there exists $p, q \in [a,b]$ such that $\tilde{f}(p) = i \circ f(p) = 0$ and $\tilde{f}(q) = i \circ f(q) = 1$. By intermediate value theorem, there exists $r \in [a,b]$ such that $\tilde{f}(r) = i \circ f(r) = 1/2$, which implies $f(r) = \frac{1}{2}$, which is a contradiction. ■

Definition 4.5 [Connected subset] A non-empty subset $S \subseteq X$ is **connected** if S with the subspace topology is connected

Equivalently, $S \subseteq X$ is connected if, whenever U, V are open in X such that $S \subseteq U \cup V$, and $(U \cap V) \cap S = \emptyset$, one can imply either $U \cap S = \emptyset$ or $V \cap S = \emptyset$.

Proposition 4.9 If $f : X \to Y$ is continuous mapping, and the subset $A \subseteq X$ is connected, then $f(A)$ is connected. In other words, the continuous image of a connected set is connected.

Proof. Suppose that $U, V \subseteq Y$ is open such that

$$f(A) \subseteq U \cup V, \quad (U \cap V) \cap f(A) = \emptyset.$$

Therefore we imply

$$A \subseteq f^{-1}(U) \cup f^{-1}(V), \quad (f^{-1}(U) \cap A) \cap (f^{-1}(V) \cap A) = \emptyset$$

By connectedness of A, either $f^{-1}(U) \cap A = \emptyset$ or $f^{-1}(V) \cap A = \emptyset$. Therefore, $f(A) \cap U = \emptyset$ or $f(A) \cap V = \emptyset$, i.e., $f(A)$ is connected. ■

Proposition 4.10 If $\{A_i\}_{i \in I}$ are connnected and $A_i \cap A_j \neq \emptyset$ for $\forall i, j \in I$, then the set $\bigcup_{i \in I} A_i$ is connected.

Proof. Suppose the function $f : \cup_{i \in I} A_i \to \{0, 1\}$ is a continuous map. Then we imply that its restriction $f|_{A_i} = f \circ i : A_i \to \{0, 1\}$ is continuous for all $i \in I$. Thus $f|_{A_i}$ is a constant for all $i \in I$. Due to the non-empty intersection of A_i, A_j for $\forall i, j \in I$, we imply f is constant. ■

Proposition 4.11 If X, Y are connnected, then $X \times Y$ is connected using product topology.

Proof. It's clear that $X \times \{y_0\}$ is connected in $X \times Y$ for fixed y_0; and $\{x_0\} \times Y$ is connected for fixed x_0.

Therefore, for fixed $y_0 \in Y$, construct $B = X \times \{y_0\}$ and $C_x = \{x\} \times Y$, which follows that

$$B \cap C_x = \{(x, y_0)\} \neq \emptyset, \forall x \in X \implies B \cup \left\{ \bigcup_{x \in X} C_x \right\} = X \times Y \text{ is connected.}$$

■

Definition 4.6 [Path Connectes] Let $(X, \mathcal{T})$ be a topological space.

1. A path connecting 2 points $x, y \in X$ is a continuous function $\tau : [0,1] \to X$ with $\tau(0) = x, \tau(1) = y$.
2. X is path-connected if any 2 points in X can be connected by a path.
3. The set $A \subseteq X$ is path-connected, if A sastisfies the condition using subspace topology.

   Or equivalently, A is path-connected if for any 2 points in X, there exists a continuous $t : [0,1] \to X$ with $t(x) \in A$ for any x, connecting the 2 points.

4.6. Wednesday for MAT4002

There will be a quiz on Monday.

Reviewing.

- Connectedness / Path-Connectedness

4.6.1. Remark on Connectedness

Proposition 4.14 All path connected spaces X are connected.

Proof. Fix any $x \in X$, for all $y \in X$, there exists a continuous mapping $p_y : [0,1] \to X$ such that

$$p_y(0) = x, \quad p_y(1) = y.$$

Consider $C_y = p_y([0,1])$, which is connected, due to proposition (4.9).

Note that $\{C_y\}_{y\in X}$ is a collection of connected sets, and for any $y, y' \in X$, $C_y \cap C_{y'} \ni \{x\}$ is non-empty. Applying proposition (4.10), we imply $X = \cup_{y\in X} C_y$ is connected. ■

▪ **Example 4.5** 1. Exercise: if $A \subset B \subset \overline{A}$, then A is connected implies B is connected. (Hint: $U \cap A = \emptyset$ implies $U \cap \overline{A} = \emptyset$ for all open sets U in X.)

Proof. Suppose B is not connected, i.e., for any open U, V such that $B \subseteq U \cup V$ and $(U \cap V) \cap B = \emptyset$, we imply $U \cap B \neq \emptyset$ and $V \cap B \neq \emptyset$, and therefore

$$U \cap \overline{A} \neq \emptyset, \quad V \cap \overline{A} \neq \emptyset$$

which implies

$$U \cap A \neq \emptyset, \quad V \cap A \neq \emptyset$$

which contradicts to the connectedness of A. ■

2. The converse of proposition (4.14) may not be necessarily true. Consider the so-called **Topologist's comb** example:

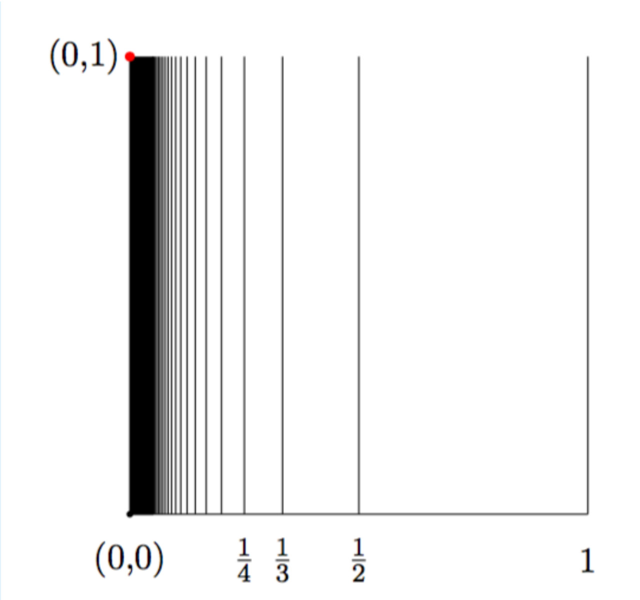


Figure 4.1: Connected space X but not path-connected

Here we construct a connected space $X \subseteq \mathbb{R}^2$ but not path-connected shown in Fig (4.1), i.e., the union of the interval $[0,1]$ together with vertical line segments from $(1/n,0)$ to $(1/n,1)$ and the single point $(0,1)$.

$$X = ([0,1] \times \{0\}) \cup \bigcup_{n\geq 1}(\{1/n\} \times [0,1]) \cup (0,1).$$

(a) Firstly, X is not path-connected. We show that there is no path in X links $(0,1)$ to any other point, i.e., for continuous mapping $p : [0,1] \to X$ with $p(0) = (0,1)$, we may imply $p(t) = (0,1)$ for any t.

Define

$$A = \{t \in [0,1] \mid p(t) = (0,1)\}.$$

We claim that $A = [0,1]$, i.e., suffices to show A is both open and closed in $[0,1]$:

i. The set $A = p^{-1}((0,1))$ is nonempty and closed, since the pre-image of a closed set is closed as well.

ii. The set A is open: choose $t_0 \in A$. By continuity of p, there exists $\delta > 0$ such that

$$\|p(t) - (0,1)\| = \|p(t) - p(t_0)\| < \frac{1}{2}, \quad t \in [0,1] \cap (t_0 - \delta, t_0 + \delta).$$

Since there is no point on the x-axis with the distance $1/2$ to the point $(0,1)$, we imply $p(t)$ is not on the x-axis when $t \in [0,1] \cap (t_0 - \delta, t_0 + \delta)$. Therefore, the x-coordinate of $p(t)$ is either 0 or of the form $1/n$.

It suffices to show the open interval $I := [0,1] \cap (t_0 - \delta, t_0 + \delta)$ is in A. Define the composite function $f = x \circ p : I \to \mathbb{R}$, where the mapping $x : \mathbb{R}^2 \to \mathbb{R}$ is defined as $(a,b) \mapsto a$. Note that I is connected, we imply $f(I)$ is connected, and $f(I)$ belongs to $\{0\} \cup \{1/n\}$.

The only nonempty connected subset of $\{0\} \cup \{1/n\}$ is a single point (left as exercise), and therefore $f(I)$ is a single point. Since $f(t_0) = 0$,we imply $f(I) = \{0\}$, i.e., $I \subseteq A$. Therefore A is open.

4.6.2. Compactness

Compact set in X is used to generalize "closed and bounded" in $\mathbb{R}^n$.

Definition 4.11 Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{U} = \{U_i \mid i \in I\}$ of open sets is an open cover of X if

$$X = \bigcup_{i \in I} U_i$$

A subcover of $\mathcal{U}$ is a subfamily

$$\mathcal{U}' = \{U_j \mid j \in J\}, \quad J \subseteq I$$

such that $\bigcup_{j \in J} U_j = X$.

If J has finitely many elements, we say $\mathcal{U}'$ is a finite subcover of X.

We say X is compact if any open cover of X has a finite subcover.

R If $A \subseteq X$ has a subspace topology. then A is compact iff for any open collection of open sets (in X) $\{U_i\}$ such that $A \subseteq \bigcup_{i\in I} U_i$, there exists a fintie subcover $A \subseteq \bigcup_{k=1}^n U_{i_k}$.

Proposition 4.15 Let X be a topological space. The followings are equivalent:

1. The space X is compact
2. If $\{V_i \mid i \in I\}$ is a collection of closed subsets in X such that

$$\bigcap_{j\in J} V_j \neq \emptyset, \quad \text{for all finite } J \subseteq I,$$

then $\cap_{i\in I} V_i \neq \emptyset$.

Compactness is an **intrisical** property, i.e., we do not need to worry about which underlying space for this definition.

▪ **Example 4.6** 1. $X \subseteq \mathbb{R}^n$ is compact iff X is closed and bounded. (Heine-Borel)

2. Let $K \subseteq \mathbb{R}^n$ be compact, then define the set

$$C(K) = \{\text{all continuous mapping } f : K \to \mathbb{R}\}$$

Note that the d_∞ metric associated with $C(K)$, say $\|f\|_\infty = \sup_{k\in K} f(k)$, is well-defined.

Under the metric space $(C(K), d_\infty)$, any $\mathcal{J} \subseteq C(K)$ is compact, if and only if $\mathcal{J}$ is closed, bounded, and equi-continuous. (Aresul-Ascoli)

Therefore, we can see that the compactness is not equivalent to the closedness together with boundedness. ▪

Proposition 4.16 Let X be a compact space, then all closed subset $A \subseteq X$ are compact.

Proof. Let $\{V_i \mid i \in I\}$ be a collection of closed subsets in A such that

$$\cap_{j\in J} V_j \neq \emptyset, \quad \text{for any finite } J \subseteq I.$$

As A is closed in X, we imply V_j is closed in X.

Due to the compactness of X and proposition (4.15), we imply

$$\cap_{i\in I} V_i \neq \emptyset$$

By the reverse direction of proposition (4.15), we imply A is compact. ■

R Now consider the reverse direction of proposition (4.16), i.e., are all compact subsets $K \subseteq X$ closed in X?

In general, the converse does not hold. Note that $K = \{x\}$ is compact for any topology X. However, there are some topologies such that $\{x\}$ is closed.

In order to obtain the converse of proposition (4.16), we need to obtain another **separation axiom**:

Proposition 4.17 Let X be Hausdorff, $K \subseteq X$ be compact, and $x \in X \setminus K$. Then there exists open $U, V \subseteq X$ such that $U \cap V = \emptyset$ and

$$U \cap V = \emptyset, \quad K \subseteq U, \quad x \in V.$$

Proof. Let $k \in K$, then by Hausdorffness, there exists open $U_k \ni k, V_k \ni x$ such that $U_k \cap V_k = \emptyset$. Therefore, $\{U_k\}_{k\in K}$ forms an open cover of K. By compactness of K, $\{U_{k_i}\}_{i=1}^n$ covers K. Constructing the set

$$U := \bigcup_{i=1}^n U_{k_i}, \quad V = \bigcap_{i=1}^n V_{k_i}$$

gives the desired result. ■

By making use of this separation axiom, we obtain the converse of proposition (4.16):

Corollary 4.3 All compact K in Hausdorff X is closed.

Proof. For $\forall x \in X \setminus K$, by proposition (4.17) there exists open V such that $x \in V \subseteq X \setminus K$, and therefore $X \setminus K$ is open. ■

5.3. Monday for MAT4002

5.3.1. Continuous Functions on Compact Space

Proposition 5.3 Let $f : X \to Y$ be continuous function on topological spaces, with $A \subseteq X$ compact. Then $f(A) \subseteq Y$ is compact.

Proof. Let $\{U_i \mid i \in I\}$ be an open cover of $f(A)$, i.e.,

$$f(A) \subseteq \bigcup_{i \in I} U_i, \quad U_i \in \mathcal{T}_Y$$

It follows that $\{f^{-1}(U_i) \mid i \in I\}$ is an open cover of A:

$$A \subseteq f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i)$$

By the compactness of A, there exists finite subcover of A:

$$A \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k}),$$

which implies the constructed finite subcover of $f(A)$:

$$\begin{aligned} f(A) &\subseteq f(\cup_{k=1}^{n} f^{-1}(U_{i_k})) \\ &= \bigcup_{k=1}^{n} U_{i_k} \end{aligned}$$

■

Corollary 5.2
1. Suppose that X is compact, and the mapping $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is closed and bounded, i.e., there exists $m, M \in X$ such that $f(m) \le f(x) \le f(M)$, $\forall x \in X$.
2. Suppose moreover that X is connected, then

$$f(X) = [f(m), f(M)].$$

Theorem 5.2 The space X, Y are compact iff $X \times Y$ is compact under product topology.

Proof. 1. *Sufficiency:* Given that $X \times Y$ is compact, consider the projection mapping (which is continuous):

$$\begin{cases} P_X : X \times Y \to X \\ P_Y : X \times Y \to Y \end{cases}$$

By applying proposition (5.3), $P_X(X \times Y) = X, P_Y(X \times Y) = Y$ are both compact.

2. *Necessity:* Suppose that $\{W_i\}_{i \in I}$ is an open cover of $X \times Y$. Each open set W_i can be written as:

$$W_i = \bigcup_{j \in \mathcal{J}_i} U_{ij} \times V_{ij}, \quad U_{ij} \in \mathcal{T}_X, V_{ij} \in \mathcal{T}_Y.$$

It follows that

$$X \times Y = \bigcup_{(i,j) \in K} U_{ij} \times V_{ij}, \quad K = \{(i,j) \mid i \in I, j \in \mathcal{J}_i\}$$

Therefore, it suffices to show $\{U_{ij} \times V_{ij} \mid (i,j) \in K\}$ has a finite subcover of $X \times Y$.

- Note that $X \times \{y\} \subseteq \bigcup_{(i,j) \in K} U_{ij} \times V_{ij}$ is compact for each $y \in Y$, which implies there exists finite $S_y \in K$ such that

$$X \times \{y\} \subseteq \bigcup_{s \in S_y} U_s \times V_s$$

- w.l.o.g., assume that $y \in V_s, \forall s \in S_y$, since we can remove the $U_s \times V_s$ such that $y \notin V_s$. Define the set $V_y := \cap_{s \in S_y} V_s$, which is an open set containing y. We imply $\{V_y\}_{y \in Y}$ forms an open cover of Y. By the compactness of Y,

$$\{V_{y_1}, \dots, V_{y_m}\}$$

forms a finite subcover of Y.

- For each $\ell = 1, \dots, m$,

$$X \times \{y_\ell\} \subseteq \bigcup_{s \in S_{y_\ell}} U_s \times V_s$$

Note that for any $(x, y) \in X \times Y$, there exists $\ell \in \{1, \dots, m\}$ such that $y \in V_{y\ell}$, i.e., $y \in V_s$ for $\forall s \in S_{y\ell}$. Therefore,

$$X \times Y = \bigcup_{\ell=1}^{m} \bigcup_{s \in S_{y_\ell}} U_s \times V_s$$

Now pick

$$I' = \{i \in I \mid (i, j) \in \cup_{\ell=1}^{m} S_{y\ell}\},$$

we imply $X \times Y = \bigcup_{i' \in I'} W_i$ and I' is finite.

■

Theorem 5.3 Suppose that X is compact, Y is Hausdorff, $f : X \to Y$ is continuous, bijective, then f is a **homeomorphism**.

Proof. It suffices to show f^{-1} is continuous. Therefore, it suffices to show $(f^{-1})^{-1}(V)$ is closed, given that V is closed in X:

Let $V \subseteq X$ be closed. Then V is compact, which implies $f(V)$ is compact. Since $f(V) \subseteq Y$ is Hausdorff, we imply $f(V)$ is compact, i.e., $f(V)$ is closed. ■

5.6. Wednesday for MAT4002

5.6.1. Remarks on Compactness

Theorem 5.5 X is compact, Y is Hausdorff, $f : X \to Y$ is continuous and bijective. Then X is **homeomorphic** to Y

Corollary 5.3 If X is compact, Y is Hausdorff, $f : X \to Y$ is injective and continous, then $f : X \to f(X)$ is homeomorphisc.

■ **Example 5.7** Here we give another proof for the fact that $S^1 \times S^1$ is homeomorphic to donut. Construct the mapping

$$\begin{aligned} f: \quad & S^1 \times S^1 \to \mathbb{R}^3 \\ \text{with} \quad & (e^{i\theta}, e^{i\phi}) \mapsto ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta) \quad (R > r > 0) \end{aligned}$$

Note that:

- $X = S^1 \times S^1$ is compact, $\mathbb{R}^3$ is Hausdorff;
- f is continuous and injective.
- $f(S^1 \times S^1)$ is a "donut".

Therefore, we conclude that $S^1 \times S^1$ is homeomorphic to donut in $\mathbb{R}^3$. ■

Definition 5.6 [Sequential Compactness] A topological space X is **sequentially compact** if every sequence in X has a convergent sub-sequence. ■

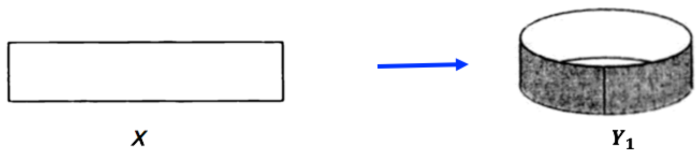
In $\mathbb{R}^n$, the compactness is equivalent to sequential compactness. The same goes for any metric space (X, d). (Check notes for MAT3006)

However, compactness and sequential compactness is different for topological spaces in general.

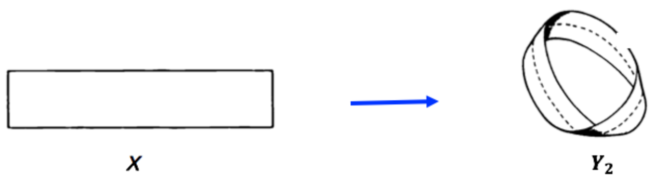
5.6.2. Quotient Spaces

Motivation. Just like product space and disjoint union, we give another way to construct new topological spaces from some old ones. This new way of construction is by gluing some special pieces from old topological spaces together.

Idea. Let $X = [0,1] \times [0,1]$ (just like a paper on a plane), we want to glue the leftmost edge with the rightmost edge to form a cylinder Y_1, as shown below:



If we give a half-twist to the strip before glue the ends together, we will get the **Moebius stripe** Y_2 shown below:



Interestingly, the first topology Y_1 has two sides, while the second has only one side.

5.6.2.1. Equivalence Relations and partitions

Definition 5.7 [Equivalence Relation] The **equivalence** relation on a set X is a relation $\sim$ such that

1. (Reflexive): $x \sim x, \forall x \in X$
2. (Symmetric): $x \sim y$ implies $y \sim x$
3. (Transitive): $x \sim y$ and $y \sim z$ implies $x \sim z$.

■ **Example 5.8** 1. Let $X = V$ be a vector space, and $W \leq V$ be a vector subspace. Define $\boldsymbol{v}_1 \sim \boldsymbol{v}_2$ if $\boldsymbol{v}_1 - \boldsymbol{v}_2 \in W$.

(The well-definedness is left as exercise).

2. (Mobius Stripe): Let $X = [0,1] \times [0,1]$. We define $(x_1, y_1) \sim (x_2, y_2)$ if

- $x_1 = x_2, y_1 = y_2$; (e.g., $(0.5, 0.6) \sim (0.5, 0.6)$) or
- $x_1 = 0, x_2 = 1$, and $y_1 = 1 - y_2$ (e.g., $(0, 1/4) \sim (1, 3/4)$)
- $x_1 = 1, x_2 = 0$, and $y_1 = 1 - y_2$ (e.g., $(1, 3/4) \sim (0, 1/4)$)

■

Definition 5.8 [Partition] Let X be a nonempty set. A **partition** $\mathcal{P} = \{p_i \mid i \in I\}$ of X is a collection of subsets such that

1. $P_i \subseteq X$ is non-empty
2. $P_i \cap P_j = \emptyset$ if $i \neq j$
3. $\bigcup_{i \in I} P_i = X$

■

R Given a partition $\mathcal{P} = \{p_i \mid i \in I\}$, we can define an equivalence relation $\sim$ on X by setting

$$x \sim y \quad \text{whenever } x, y \in p_i, \text{ for some } i \in I$$

For example, if $X = [0,1] \times [0,1]$, then

$$X = \{(x,y)\}_{x \in (0,1), y \in [0,1]} \cup \{(1,y), (0, 1-y)\}_{y \in [0,1]}$$

gives a partition on X. This gives the same equivalence relation as in part (2) in example (5.8).

Conversely, given an equivalence relation $\sim$, we could form a corresponding partition of X. This kind of partition is called the equivalence class:

Definition 5.9 [Equivalence Class] Let X be a set with equivalence relation $\sim$. The equivalence class of an element $x \in X$ is

$$[x] := \{y \in X \mid x \sim y\}.$$

Proposition 5.8 The collection of all $[x]$ in $X/\sim$ gives a partition on X.

Consider the equivalence class defined in part (1) in example (5.8). The equivalence class has the form

$$[\boldsymbol{v}] = \{\boldsymbol{u} \in V \mid \boldsymbol{v} - \boldsymbol{u} \in W\} := \boldsymbol{v} + W.$$

Therefore, the equivalence class is a generalization of the **coset** in linear algebra. Similarly, we define the set of generalized cosets as **quotient space**.

Definition 5.10 The collection of all equivalence classes is called the **quotient space**, denoted as $X/\sim$, i.e.,

$$X/\sim = \{[x] \mid x \in X\}.$$

Example 5.9

1. Consider part (1) in example (5.8) again. The quotient space $V/\sim$ reduces to the V/W in linear algebra:

$$V/\sim = \{[\boldsymbol{v}] \mid \boldsymbol{v} \in V\} = \{\boldsymbol{v} + W \mid \boldsymbol{v} \in V\} = V/W.$$

2. Consider part (2) in example (5.8) again. Then $X/\sim$ essentially forms the **Mobius band**, e.g.,

$$[(1/2, 1/2)] = \{x \mid (1/2, 1/2) \sim x\} = \{(1/2, 1/2)\}$$

$$[(1, 3/4)] = \{x \mid x \sim (1, 3/4)\} = \{(1, 3/4), (0, 1/4)\}$$

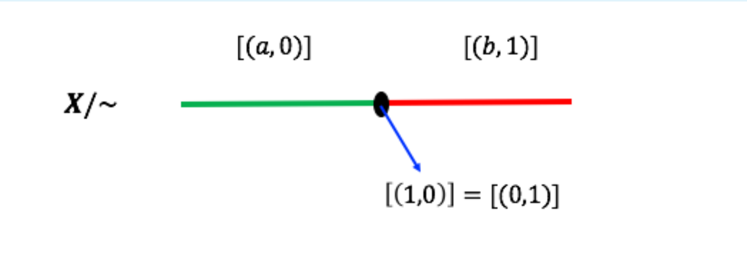
■ **Example 5.10** Consider $X = [0,1] \sqcup [0,1]$, i.e.,

$$X = ([0,1] \times \{0\}) \cup ([0,1] \times \{1\})$$

Take a partition on X by

$$\{(a,0)\}_{0 \le a < 1} \cup \{(b,1)\}_{0 < b \le 1} \cup \{(1,0),(0,1)\}$$

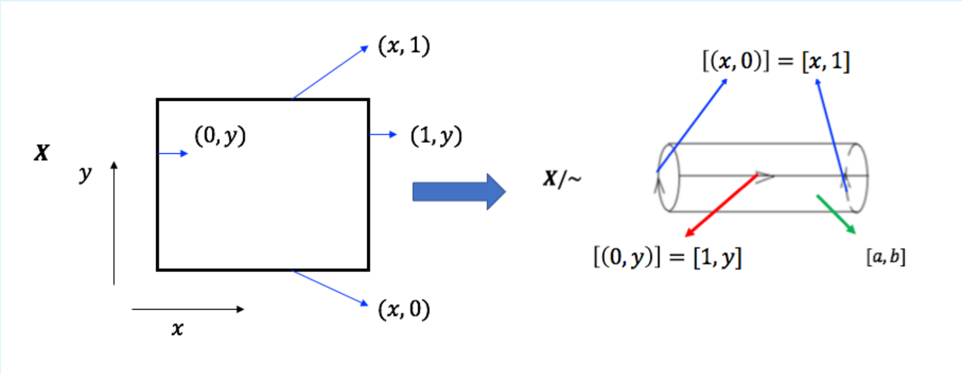
As a result, the corresponding quotient space is plotted below:



■ **Example 5.11** Comes from $X = [0,1] \times [0,1]$ with partition

$$\{(a,b)\}_{0<a<1;0<b<1} \cup \{(x,0),(x,1)\}_{0 \le x \le 1} \cup \{(0,y),(1,y)\}_{0<y<1}$$

The corresponding quotient space is plotted below:



Proposition 5.9 Let $(X, \mathcal{T})$ be topological space, with the equivalence relation. Define the canonical projection map

$$\begin{aligned} p: &\quad X \to X/{\sim} \\ \text{with} &\quad x \mapsto [x] \end{aligned}$$

Define a collection of subsets $\tilde{\mathcal{T}}$ on $X/{\sim}$ by:

$$U \subseteq X/{\sim} \text{ is in } \tilde{\mathcal{T}} \text{ if } p^{-1}(U) \text{ is in } \mathcal{T}.$$

Then $\tilde{\mathcal{T}}$ is a topology for $X/{\sim}$, called **quotient topology**.

6.3. Monday for MAT4002

6.3.1. Quotient Topology

Now given a topologcal space X and an equivalence relation $\sim$ on it, our goal is to construct a topology on the space $X/\sim$.

Proposition 6.1 Suppose $(X,\mathcal{T})$ is a topological space, and $\sim$ is an equivalene relation on X. Define the canonical projection map:

$$\begin{array}{ll} p: & X \to X/\sim \\ \text{with} & x \to [x] \end{array},$$

which assigns each point $x \in X$ into the equivalence class $[x]$. Then define a family of subsets $\tilde{\mathcal{T}}$ on $X/\sim$ by:

$$\tilde{U} \subseteq X/\sim \ \text{ is in } \tilde{\mathcal{T}} \text{ if } p^{-1}(\tilde{U}) \text{ is in } \mathcal{T}$$

Then $\tilde{\mathcal{T}}$ is a topology for $X/\sim$, called the **quotient topology**, and $(X/\sim,\tilde{\mathcal{T}})$ is called the quotient space, and $p: X \to X/\sim$ is called the **natural map**.

Proof.

1. $p^{-1}(X/\sim) = X \in \mathcal{T}$ and $p^{-1}(\emptyset) = \emptyset \in \mathcal{T}$, which implies $X/\sim \in \tilde{\mathcal{T}}$ and $\emptyset \in \tilde{\mathcal{T}}$.
2. Suppose that $\tilde{U},\tilde{V} \in \tilde{\mathcal{T}}$, then we imply

$$p^{-1}(\tilde{U}),p^{-1}(\tilde{V}) \in \mathcal{T} \implies p^{-1}(\tilde{U} \cap \tilde{V}) \in \mathcal{T},$$

   i.e., $\tilde{U} \cap \tilde{V} \in \tilde{\mathcal{T}}$.
3. Following the similar argument in (2), and the relation

$$p^{-1}\left(\bigcup \tilde{U}_i\right) = \bigcup p^{-1}(\tilde{U}_i),$$

   we conclude that $\tilde{T}$ is closed under countably union.

The proof is complete. ∎

R

1. The proposition (6.1) claims that $\tilde{U}$ is open in $X/\sim$ iff $p^{-1}(\tilde{U})$ is open in X. The general question is that, does $p(U)$ is open in $X/\sim$, given that U is open in X? This may not necessarily hold. (See example (6.4)) In general $p^{-1}(p(U))$ is strictly larger than U, and may not be necessarily open in X, even when U is open.
2. By definition, we can show that p is continuous.

To fill the gap on the question shown in the remark, we consider the notion of the open mapping:

Definition 6.3 [Open Mapping] A function $f : X \to Y$ between two topological spaces is an **open mapping** if for each open U in X, $f(U)$ is open in Y.

R From the remark above, we can see that:

1. Not every continuous mapping is an open mapping
2. The canonical projection mapping p is not necessarily be an open mapping.

Example 6.4 1. The mapping $p : [0,1]\times[0,1] \to ([0,1]\times[0,1])/\sim$ sending the square to the Mobius band M is not an open mapping:

Consider the open ball $U = B_{1/2}((0,0))$ in $[0,1]\times[0,1]$. Note that $p(U)$ is open in M iff $p^{-1}(p(U))$ is open in $[0,1]\times[0,1]$. We can calculate $p^{-1}(p(U))$ explicitly:

$$p^{-1}(p(U)) = U \cup \{(1,y) \mid 1/2 \le y \le 1\},$$

which is not open.

6.3.2. Properties in quotient spaces

6.3.2.1. Closedness on $X/\sim$

Proposition 6.2 A subset $\tilde{V}$ is closed in the quotient space $X/\sim$ iff $p^1(\tilde{V})$ is closed in X, where $p : X \to X/\sim$ denotes the canonical projection mapping.

Proof. It follows from the fact that

$$p^{-1}\left((X/\sim)\setminus\tilde{V}\right) = X \setminus p^{-1}(\tilde{V})$$

∎

6.3.2.2. Isomorphism on $X/\sim$

The quotient space can be used to study other type of spaces:

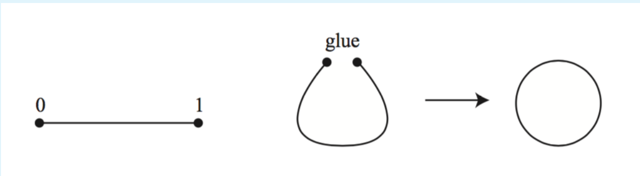
▪ **Example 6.5** Consider $X = [0,1]$. We define $x_1 \sim x_2$ if:

$$x_1 = 0, x_2 = 1, \quad \text{or} \quad x_1 = 1, x_2 = 0$$

In other words, the partition on X is given by:

$$X = \{0,1\} \cup \left(\bigcup_{x\in(0,1)} \{x\}\right)$$

The quotient space seems "glue" the endpoints of the interval $[0,1]$ together, shown in the figure below:



It is intuitive that the constructed quotient space should be homeomorphic to a circle S^1. We will give a formal proof on this fact. ▪

Proposition 6.3 Let X and Z be topological spaces, and $\sim$ an equivalence relation on X.

Let $g : X/\sim \longrightarrow Z$ be a function, and $p : X \to X/\sim$ is a projection mapping The mapping g is continuous if and only if $g \circ p : X \to Z$ is continuous.

Proof. 1. *Necessity*. Suppose that g is continuous. It's clear that p is continuos, i.e, $g \circ p : X \to Z$ is continuous.

2. *Sufficiency*. Suppose that $g \circ p : X \to Z$ is continuous. Given any open U in Z, we imply $(g \circ p)^{-1}(U) = p^{-1}g^{-1}(U)$ is open in X. By definition of the quotient topology, we imply $g^{-1}(U)$ is open in $X/\sim$. Therefore, g is continuous.

∎

R This useful lemma can be generalized into the case for generlized canonical projection mapping, called quotient mapping.

> **Definition 6.4** [Quotient mapping] A map $p : X \to Y$ between topological spaces is a **quotient mapping** if
>
> 1. p is **surjective**; and
> 2. p is continuous;
> 3. For any $U \subseteq Y$ such that $p^{-1}(U)$ is open in X, we imply U is open in Y.

The canonical projection map is clearly a quotient map. Actually, a stronger version of proposition (6.3) follows:

Proposition 6.4 Suppose that $p : X \to Y$ is a quotient map and that $g : Y \to Z$ is any mapping to another space Z. Then g is continuous iff $g \circ p$ is continuous.

Proof. The proof follows similarly as in proposition (6.3). ∎

Now we give a formal proof of the conclusion in the example (6.5):

Proof. Define the mapping

$$\begin{aligned} f : \quad & [0,1] \to S^1 \\ \text{with} \quad & t \mapsto (\cos 2\pi t, \sin 2\pi t). \end{aligned}$$

Since $f(0) = f(1)$, the function f induces a well-defined function

$$\begin{aligned} g: \quad & [0,1]/\sim \longrightarrow S^1 \\ \text{with} \quad & [t] \mapsto f(t) \end{aligned}$$

such that $f = g \circ p$, where p denotes the canonical projection mapping. Note that f is continuous. By proposition (6.3), we imply g is continuous. Furthermore,

1. Since $[0,1]$ is compact and p is continuous, we imply $p([0,1]) = [0,1]/\sim$ is compact
2. S^1 is Hausdorff
3. g is a bijection

By applying theorem(5.3), we conclude that g is a homeomorphism, i.e., $[0,1]/\sim$ and S^1 are homeomorphic.

∎

The argument in the proof can be generalized into the proposition below:

Proposition 6.5 Let $f : X \to Y$ be a surjective continuous mapping between topologcial spaces. Let $\sim$ be the equivalence relation on X defined by the partition $\{f^{-1}(y) \mid y \in Y\}$ (i.e., $f(x) = (x')$ iff $x \sim x'$). If X is compact and Y is Hausdorff, then $X/\sim$ and Y are homeomorphic.

R The proposition (6.5) is a pattern of argument we should use several times. In order to show $X/\sim$ and Y are homeomorphic, we should think up a surjective continuous mapping $f : X \to Y$ "with respect to the identifications", i.e., $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$. Therefore f will induce a well-defined function $g : X/\sim \longrightarrow Y$ such that $f = g \circ f$. Then checking the conditions in theorem(5.3) leads to the desired results.

Torus. We now study the torus in more detail.

1. Consider $X = [0,1] \times [0,1]$ and define $(s_1, t_1) \sim (s_2, t_2)$ if one of the following holds:
   - $s_1 = s_2$ and $t_1 = t_2$;
   - $\{s_1, s_2\} = \{0,1\}$, $t_1 = t_2$;

- $\{t_1, t_2\} = \{0, 1\}$ and $s_1 = s_2$;
- $\{s_1, s_2\} = \{0, 1\}$, $\{t_1, t_2\} = \{0, 1\}$

The corresponding quotient space $([0,1] \times [0,1])/\sim$ is hoemomorhpic to the 2-dimension torus $\mathbb{T}^2$.

Proof. Define the mapping $f : [0,1] \times [0,1] \to \mathbb{T}^2$ as $(t_1, t_2) \mapsto (e^{2\pi i t_1}, e^{2\pi i t_2})$.

(a) f is surjective, which also implies $\mathbb{T}^2 = f([0,1] \times [0,1])$ is compact.

(b) $\mathbb{T}^2$ is Hausdorff

(c) It's clear that $(s_1, t_1) \sim (s_2, t_2)$ implies $f(s_1, t_1) = f(s_2, t_2)$. Conversely, suppose

$$e^{2\pi i s_1} = e^{2\pi i s_2}, \quad e^{2\pi i t_1} = e^{2\pi i t_2}$$

By the familiar property of e^{ix}, we imply either $t_1 = t_2$ or $\{t_1, t_2\} = \{0, 1\}$; and either $s_1 = s_2$ or $\{s_1, s_2\} = \{0, 1\}$

By applying proposition (6.5), we conclude that $([0,1] \times [0,1])/\sim$ is homeomorphic to $\mathbb{T}^2$. ∎

2. Consider the closed disk $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$, and defube $(x_1, y_1) \sim (x_2, y_2)$ if one of the following holds:

   - $x_1 = x_2$ and $y_1 = y_2$;
   - (x_1, y_1) and (x_2, y_2) are in the boundary circle $\mathbb{S}^1$

   The corresponding quotient space $\mathbb{D}^2/\sim$ is hoemomorhpic to the 2-dimension sphere $\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$.

   Proof. Define the mapping

$$\begin{aligned} f: \quad & \mathbb{D}^2 \to \mathbb{S}^2 \\ \text{with} \quad & (0,0) \mapsto (0,0,1) \\ & (x, y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}} \sin(\pi\sqrt{x^2+y^2}), \frac{y}{\sqrt{x^2+y^2}} \sin(\pi\sqrt{x^2+y^2}), \cos(\pi\sqrt{x^2+y^2}) \right) \end{aligned}$$

   It's easy to check the conditions in proposition (6.5), and we conclude that $\mathbb{D}^2/\sim$ is hoemomorhpic to $\mathbb{S}^2$ ∎

7.3. Monday for MAT4002

7.3.1. Quotient Map

Definition 7.6 [Quotient Map] A mapping $q: X \to Y$ between topological spaces is a **quotient map** if

1. q is surjective
2. For any $U \subseteq Y$, U is open iff $q^{-1}(U)$ is open.

R

1. The canonical projection mapping $p: X \to X/\sim$ is a quotient mapping
2. We say f is an open mapping if U is open in X implies $f(U)$ is open in Y. Note that a continuous open mapping satisfies condition (2) in definition (7.6).

In proposition (6.5) we show the homeomorphism between $X/\sim$ and Y given the compactness of X and Hausdorffness of Y. Now we show the homeomorphism by replacing these conditions with the quotient mapping q:

Proposition 7.9 Suppose $q: X \to Y$ is a quotient map, and that $\sim$ is an equivalence relation on X given by the partition $\{q^{-1}(y) \mid y \in Y\}$. Then $X/\sim$ and Y are **homeomorphic**.

Proof. Construct the mapping

$$\begin{aligned} h: &\quad X/\sim \to Y \\ \text{with} &\quad h([x]) = q(x) \end{aligned}$$

Note that:

1. The mapping h is well-defined and injective.
2. Surjective is easy to shown.
3. The quotient mapping $q := h \circ p$, by definition, is continuous. By applying proposition (6.4), h is continuous.

It suffices to show h^{-1} is continuous:

- For any open $\tilde{U} \subseteq X/\sim$, it suffices to show $h(\tilde{U})$ is open in Y.

  Note that
  $$q^{-1}(h(\tilde{U})) = p^{-1}h^{-1}(h(\tilde{U})) = p^{-1}(\tilde{U}),$$
  which is open by the definition of quotient topology (check proposition (6.1)).

  Therefore, $h(\tilde{U})$ is open by (2) in definition (7.6).

■

▪ **Example 7.4** The $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle S^1:

Define the mapping
$$\begin{aligned} q: \quad & \mathbb{R} \to S^1 \\ & x \mapsto e^{2\pi i x} \end{aligned}$$

It's clear that

1. q is a continuous open mapping (why?)
2. q is surjective

Therefore, $\mathbb{R}/\sim \cong S^1$, provided that $x \sim y$ iff $q(x) = q(y)$, i.e., $x - y \in \mathbb{Z}$. Therefore,
$$\mathbb{R}/\mathbb{Z} \cong S^1$$

7.3.2. Simplicial Complex

Combinatorics is the slums of topology. — J. H. C. Whitehead

The idea is to build some new spaces from some "fundamental" objects. The combinatorialists often study topology by the combinatorics of these fundamental objects. First we define what are the "fundamental" objects:

Definition 7.7 [n-simplex] The standard n-simplex is the set

$$\Delta^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \middle| x_i \geq 0, \forall i \text{ and } \sum_{i=1}^{n+1} x_i = 1 \right\}$$

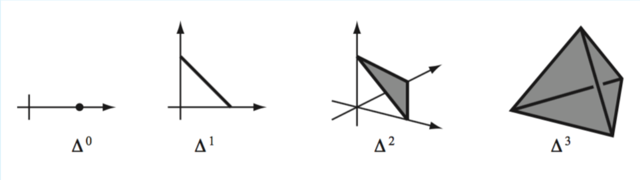


Figure 7.1: Simplices on $\mathbb{R}^2$ are the triangles, so you may consider simplexes as the "triangles" in general spaces

1. The non-negative integer n is the **dimension** of this simplex
2. Its **vertices**, denoted as $V(\Delta^n)$, are those points $(x_1, \ldots, x_{n+1})$ in Δ^n such that $x_i = 1$ for some i.
3. For each given non-empty $\mathcal{A} \subseteq \{1, \ldots, n+1\}$, its **face** is defined as

$$\{(x_1, \ldots, x_{n+1}) \in \Delta^n \mid x_i = 0, \ \forall i \notin \mathcal{A}\}$$

   In particular, Δ^n is a face of itself
4. The **inside** of Δ^n is

$$\text{inside}(\Delta^n) := \{(x_1, \ldots, x_{n+1}) \in \Delta^n \mid x_i > 0, \forall i\}$$

   In particular, the inside of Δ^0 is Δ^0.

Definition 7.8 [Face Inclusion] A face inclusion of Δ^m into Δ^n $(m < n)$ is a function $\Delta^m \to \Delta^n$ which comes from the restriction of an **injective linear** map $f : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ that maps vertices in Δ^m into vertices in Δ^n.

For example, the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined below is a face inclusion:

$$f(1,0) = (0,1,0), \quad f(0,1) = (0,0,1).$$

R Any injection mapping from $\{1,\dots,m+1\} \to \{1,\dots,n+1\}$ gives a face inclusion $\Delta^m \to \Delta^n$, and vice versa.

Motivation. Now we build new spaces by making use of simplices. This new space is called the **abstract complex**. If a simplex is a part of the complex, so are all its faces.

Definition 7.9 [Abstract Simplicial Complex] An (abstract) **simplicial complex** is a pair $K = (V, \Sigma)$, where V is a set of vertices and Σ is a collection of non-empty finite subsets of V (simplices) such that

1. For any $\boldsymbol{v} \in V$, the 1-element set $\{\boldsymbol{v}\}$ is in Σ
2. If σ is an element of Σ, then so is any non-empty subset of σ.

For example, if $V = \{1,2,3,4\}$, then

$$\Sigma = \{\{1\},\{2\},\{3\},\{4\},\{1,3,4\},\{2,4\},\{1,3\},\{3,4\},\{1,4\}\}$$

We can associate to an abstract simplicial complex K a topological space $|K|$, which is called its **geometric realization**:

Definition 7.10 [Topological Realization] The **topological realization** of $K = (V, \Sigma)$ is a topological space $|K|$ (or denoted as $|(V,\Sigma)|$), where

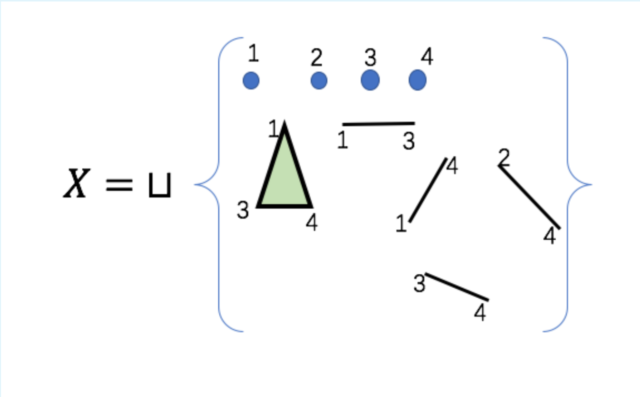
1. For each $\sigma \in \Sigma$ with $|\sigma| = n+1$, take a copy of n-simplex and denote it as Δ_σ
2. Whenever $\sigma \subset \tau \in \Sigma$, identify Δ_σ with a face of Δ_τ through face inclusion.

Or equivalently, $|K|$ is a quotient space of the disjoint union

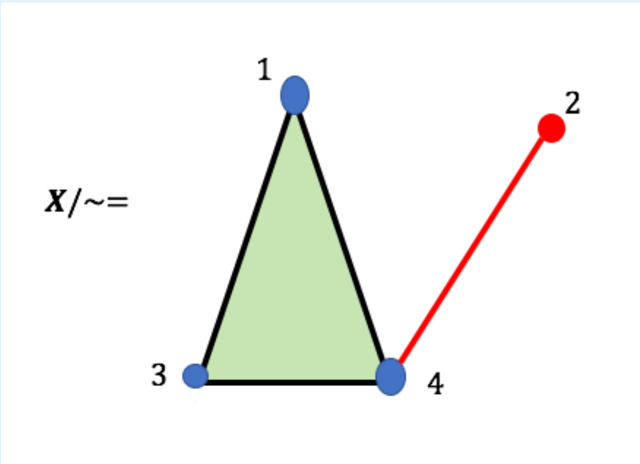
$$\coprod_{\sigma\in\Sigma}\sigma$$

by the equivalence relation which identifies a point $y\in\sigma$ with its image under the face inclusion $\sigma\to\tau$, for any $\sigma\subset\tau$.

Example 7.5 Take



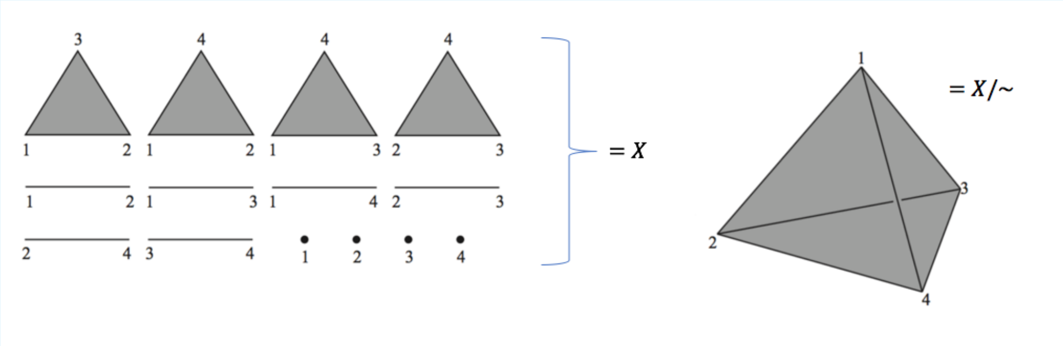
As a result,



■ **Example 7.6** Take $V = \{1,2,3,4\}$ and

$$\Sigma = \{\text{all subsets of } V \text{ except } V\}$$

As shown in the figure below, $|(V,\Sigma)| = \Delta^3$:



Definition 7.11 [Triangulation] A **triangulation** of a topological space X is a simplicial complex $K = (V,\Sigma)$ together with a choice of homeomorphism $|K| \to X$.

■ **Example 7.7** The triangulation of $S^1 \times S^1$ can be realized by using nine vertices given below:

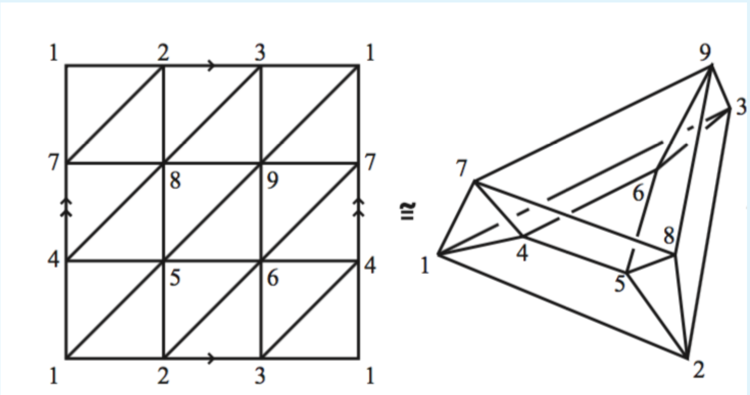


Figure 7.2: The quotient space $|K| := X/\sim$

(Try to identify X)

7.5. Wednesday for MAT4002

7.5.1. Remarks on Triangulation

Consider the simplical complex $K = (V, \Sigma)$ with

$$V = \{1,2,3,4,\dots,9\}, \quad \Sigma = \begin{cases} \text{9 subsets with 1 element} \\ \text{27 subsets with 2 elements} \\ \text{18 subsets with 3 elements} \end{cases}$$

We start to build the topological realization of K with 9 **0**-simplicies, 27 **1**-simplicies, and 18 **2**-simplicies. The identification of them is as follows:

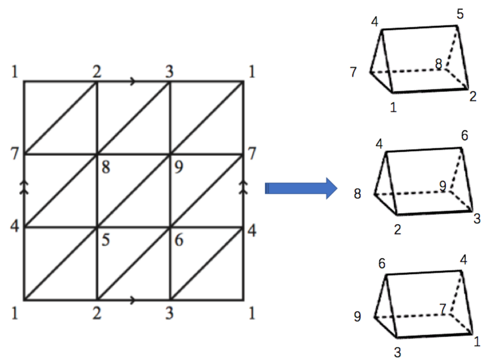


Figure 7.3: Step 1: Identify 3 columns separately, i.e., identify $\{1,7,4,1,2,8,5,2\}$, $\{2,8,5,2,3,9,6,3\}$, and $\{3,9,6,3,1,7,4,1\}$.

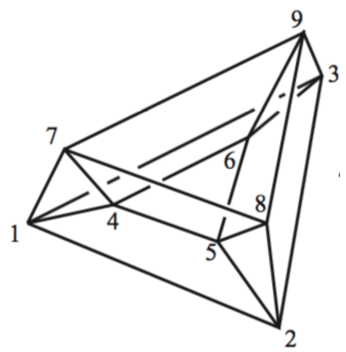
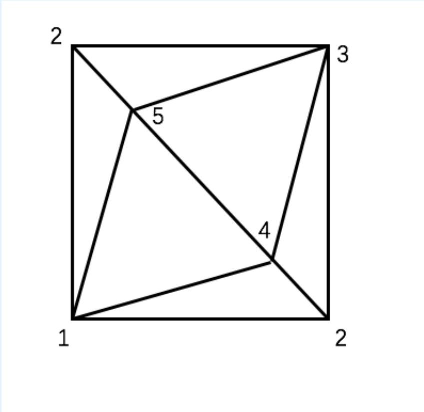


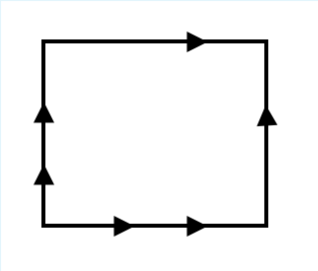
Figure 7.4: Step 2: "gluing" these three prisms in the figure above together.

Question: why K is homeomorphic to the torus?

■ **Example 7.10** Consider the simplicial complex (V,Σ) described below:

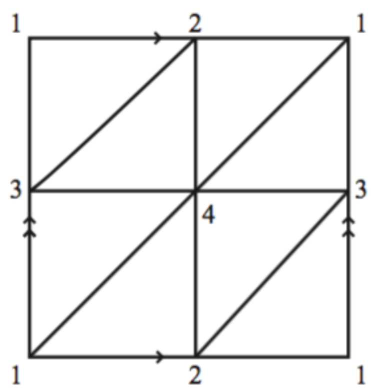


The $|(V,\Sigma)|$ is homeomorphism to the quotient space S^1 plotted below

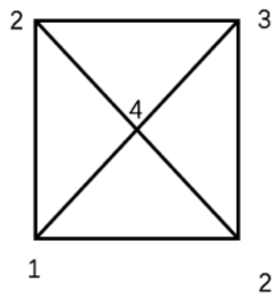


■

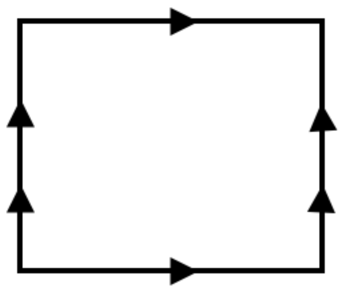
Furthermore, can we build a triangulation of the tours using fewer simplices? The answer is no. Consider the figure below: at the bottom edge of this square, there are two 1-simplicies lablled $\{1,2\}$, which cannot happen in a tours.



Interesting question: does the triangulation of the Fig. (7.1a) below leads to S^2?



(7.1a): The simplicial complex (V,Σ)



(7.1b): Quotient Space of S^2

Answer: No. Since the 2-simplex $\Delta_{\{2,3,4\}}$ appears twice in the Fig. (7.1a), the triangulation of this figure means that we need to stick the top triangle and the right triangle together, which contradicts to the structure of the quotient space S^2 shown in Fig. (7.1b).

The simplicial complex gives us another way to study X, i.e., it suffices to study (V,Σ) such that $|(V,\Sigma)| \cong X$. The question is that can we distinguish $X = S^1 \times S^1$ and $Y = S^2$? In other words, can we distinguish the difference of corresponding topological realizations?

Theorem 7.2 — **Euler's Formula.** Suppose that $|(V_1,\Sigma_1)| \cong |(V_2,\Sigma_2)|$, then

$$\begin{aligned}&\sum_{i=1}^{\infty}(-1)^i\,(\text{number of subsets in } \Sigma_1 \text{ with } (i+1)\text{-element})\\ &= \sum_{i=1}^{\infty}(-1)^i\,(\text{number of subsets in } \Sigma_2 \text{ with } (i+1)\text{-element})\end{aligned}$$

From previous examples we can see that $\mathcal{X}(S^2) = 5 - 9 + 6 = 2$ and $\mathcal{X}(S^1 \times S^1) = 9 - 27 + 18 = 0$, which implies

$$S^2 \ncong S^1 \times S^1.$$

7.5.2. Simplicial Subcomplex

Definition 7.13 [Simplicial Subcomplex] A subcomplex of a simplicial complex $K = (V, \Sigma)$ is a simplicial complex $K' = (V', \Sigma')$ such that

$$V' \subseteq V, \quad \Sigma' \subseteq \Sigma$$

Proposition 7.13 Suppose K' is subcomplex of K, then $|K'|$ is closed in $|K|$.

Proof. Suppose that D is the disjoint union of all the simplicial complex forming $|K|$. (note that the number of component in D is $|\Sigma|$)

Consider the canonical projection mapping $D \to |K|$. Observe that $p^{-1}(|K'|)$ precisely equals to $\coprod_{\sigma' \in \Sigma'} \sigma'$, which is closed in D. By definition of quotient topology, $|K'|$ is also closed. ∎

Definition 7.14 [Subcomplex spanned by vertices] Let $K = (V, \Sigma)$ be a simplicial complex and $V' \subseteq V$. Then the subcomplex spanned by V' is (V', Σ') such that

- V' denotes the vertex set.
- the simplices Σ' is given by
$$\{\sigma \in \Sigma \mid \sigma \subseteq V'\}$$

Definition 7.15 [Link and Star] Let $(V, \Sigma) = K$ be simplicial complex

- The **link** of $\boldsymbol{v} \in V$, denoted as $\mathrm{lk}(\boldsymbol{v})$ is the sub-complex with
  - vertex set
  $$\{\boldsymbol{w} \in V \setminus \{\boldsymbol{v}\} \mid \{\boldsymbol{v}, \boldsymbol{w}\} \in \Sigma\}$$
  - simplicies
  $$\{\sigma \in \Sigma \mid \boldsymbol{v} \notin \sigma \text{ and } \sigma \cup \{\boldsymbol{v}\} \in \Sigma\}$$

- The star of $\boldsymbol{v}$ (denoted as $\mathrm{st}(\boldsymbol{v})$) is

$$\bigcup\{\text{inside}(\sigma) \mid \sigma \in \Sigma, \boldsymbol{v} \in \sigma\}$$

Proposition 7.14 $\mathrm{st}(\boldsymbol{v})$ is open and $\boldsymbol{v} \in \mathrm{st}(\boldsymbol{v})$.

Proof. Omitted. ■

In fact, $|K| \setminus \mathrm{st}(\boldsymbol{v})$ is the simplicial subcomplex spanned by V.

7.5.3. Some properties of simplicial complex

Proposition 7.15 Suppose that $K = (V, \Sigma)$, where V is finite. Then $|K|$ is compact.

Proof. The mapping $p : D \to |K|$ is a canonical projection mapping, which is continuous; and D (the finite disjoint union of Δ_σ's) is compact.

Therefore, $p(D) = |K|$ is compact. ■

Proposition 7.16 For any simplicial complex $K = (V, \Sigma)$, where V is finite, there is a continuous injection

$$f : |K| \to \mathbb{R}^n \quad \text{for some } n$$

Proof. Let $K' = (V, \Sigma')$, where $\Sigma' =$ power set of V. Then

$$|K'| = \Delta^{|V|-1} \subseteq \mathbb{R}^{|V|}$$

Consider the inclusion

$$i : |K| \to |K'|$$

which comes from the following:

1. Consider the $D := \coprod_{\sigma \in \Sigma} \Delta_\sigma$ and $D' = \coprod_{\sigma' \in \Sigma'} \Delta_{\sigma'}$ in (V, Σ) and (V, Σ')
2. Construct the mapping $\tilde{i} : D \hookrightarrow D' \xrightarrow{p'} |K|$.

3. The mapping $\tilde{i}$ descends to $i : D/\sim \longrightarrow |K'|$ (try to write down the detailed mapping), which is continuous and injective.

Therefore, $|K| \hookrightarrow |K'|$, i.e., $|K| \hookrightarrow \mathbb{R}^n$. The proof is complete.

∎

8.5. Wednesday for MAT4002

Reviewing. We can construct a continuous injection from $|K|$ to $|K'|$, where $K=(V,\Sigma)$ is a simplicial complex, and $K'=(V',\Sigma')$ is its subcomplex:

Let $D_\Sigma := \coprod_{\sigma\in\Sigma}\sigma$ and $D_{\Sigma'} := \coprod_{\sigma'\in\Sigma'}\sigma'$, then $|K'| = D_{\Sigma'}/\sim_{\Sigma'}$ and $|K| = D_\Sigma/\sim_\Sigma$, which follows that

$$f : D_{\Sigma'} \to D_\Sigma \xrightarrow{P} D_\Sigma/\sim_\Sigma, \quad P \text{ denotes the canonical projection mapping}$$

The whole mapping f descends to a continuous mapping

$$\tilde{f} : D_{\Sigma'}/\sim_{\Sigma'} \to D_\Sigma/\sim_\Sigma$$

The $\tilde{f}$ is injective since

$$x \sim_{\Sigma'} y \iff i(x) \sim_\Sigma i(y), \qquad \forall x,y\in D_\Sigma, \tag{8.14}$$

where i denotes the inclusion mapping.

Another way is to consider the inclusion $i : |K'| \to |K|$, which is continuous and injective as well. Note that $i(|K'|)$ is closed in $|K|$.

Proposition 8.7 For each $K=(V,\Sigma)$, and finite V, there is a continuous injection $g : |K| \hookrightarrow \mathbb{R}^n$ for some n.

Proof. Consider $K^p := (V,\Sigma^p)$, where Σ^p is the power set of V. Therefore, $|K^p| = \Delta^{|V|-1} \subseteq \mathbb{R}^{|V|}$, and K is a simplicial subcomplex of K^p, which follows that

$$l : |K'| \xrightarrow{i} |K^p| \xrightarrow{i} \mathbb{R}^{|V|}$$

The whole mapping l is an inclusion mapping from $|K'|$ to $\mathbb{R}^{|V|}$, which is continuous and injective. The proof is complete. ∎

Proposition 8.8 — Hausdorff. If $K=(V,\Sigma)$ with fintie V, then $|K|$ is Hausdorff.

Proof. Let $g : |K| \xrightarrow{l} \mathbb{R}^n$. Consider the bijective $g : |K| \to g(|K|)$, which is continuous.

Sicne $|K|$ is compact, and $g(|K|) \subseteq \mathbb{R}^n$ is Hausdorff, we imply that $|K|$ and $g(|K|)$ are homeomorphic, i.e., $|K|$ is Hausdorff. ∎

Definition 8.14 [Edge Path] An **edge path** of $K = (V, \Sigma)$ is a sequence of vertices $(v_1, \ldots, v_n), v_i \in V$ such that $\{v_i, v_{i+1}\} \in \Sigma, \forall i$.

Proposition 8.9 — **Connectedness.** Let $K = (V, \Sigma)$ be a simplicial complex. TFAE:

1. $|K|$ is connected
2. $|K|$ is path-connected
3. Any 2 vertices in (V, Σ) can be joined by an edge path, i.e., for $\forall u, v \in V$, there exists $v_1, \ldots, v_k \in V$ such that $(u, v_1, \ldots, v_k, v)$ is an edge path.

Sketch of Proof (to be revised). 1. (3) implies (2): For every $x, y \in |K|$,

$$\begin{cases} x \in \Delta_{\sigma_1} \text{ for some } \sigma_1 \in \Sigma. \\ y \in \Delta_{\sigma_2} \text{ for some } \sigma_2 \in \Sigma. \end{cases}$$

Take a path joining x to a vertex $v_1 \in \sigma_1$ and a path joining y to a vertex $v_2 \in \sigma_2$. By (3), we have a path joninig v_1 and v_2.

2. (1) implies (3): Suppose on the contrary that there is a vertex v not satisfying (3). Take V' as the set of vertexs that can be joined with v; and V'' as the set of vertexs that cannot be joinied with v.

   Then $V', V'' \neq \emptyset$. Consider K', K'' be simplicial subcomplexes of K, spanned by V' and V''. Then $|K'|, |K''|$ are disjoint, closed in $|K|$.

   $|K| = |K'| \cup |K''|$. If there exists $x \in |K| \setminus (|K'| \cup |K''|)$, then for any $\sigma \in \Sigma$ such that $x \in \Delta_\sigma$, we imply $\Delta_\sigma \not\subseteq |K'|$ or $|K''|$.

   Therefore, σ consists of vertices in both V' and V''. Then there is $v', v'' \in \sigma$ joining V' and V''.

   Therefore, there is no such x and hence $|K| = |K'| \cup |K''|$ is a disjoint union of two closed sets, i.e., not connected.

∎

8.5.1. Homotopy

Yoneda's "philosophy". To understand an object X (in our focus, X denotes topological space), we should understand functions

$$f : A \to X, \quad \text{or} \quad g : X \to B$$

One special example is to let $B = \mathbb{R}$.

There are many type of continuous mappings from X to Y. We will group all these mappings into equivalence classes.

Definition 8.15 [Homotopy] A **Homotopy** between two continuous maps $f, g : X \to Y$ is a continuous map

$$H : X \times [0,1] \to Y$$

such that

$$H(x,0) = f(x), \quad H(x,1) = g(x)$$

If such H exists, we say f and g are **homotopic**, denoted as $f \simeq g$.

■ **Example 8.9** Let $Y \subseteq \mathbb{R}^2$ be a convex subset. Consider two continuous maps $f : X \to Y$ and $g : X \to Y$. They are always homotopic since we can define the homotopy

$$H(x,t) = tg(x) + (1-t)f(x)$$

Proposition 8.10 Homotopy is an equivalent relation.

Proof.
1. Let $f : X \to Y$ be any continuous map. Then $f \simeq f$: we can define a homotopy $H(x,t) = f(x), \forall 0 \le t \le 1$.
2. Suppose $f \simeq g$, i.e., H is a homotopy between f and g, then $g \simeq f$: Define the mapping $H'(x,t) = H(x, 1-t)$, then

$$H'(x,0) = g(x), \quad H'(x,1) = f(x)$$

3. Let $f,g,h : X \to Y$ be three continuous maps. If f and g are homotopic and g and h are homotopic, then f and h are homotopic:

   Let $H : X \times [0,1] \to Y$ be a continuous map such that

$$H(x,0) = f(x), H(x,1) = g(x);$$

   $K : X \times [0,1] \to Y$ be a continuous map such that

$$K(x,0) = g(x), K(x,1) = h(x).$$

   Define a function $J : X \times [0,1] \to Y$ by

$$J(x,t) = \begin{cases} H(x,2t), & 0 \le t \le 1/2 \\ K(x,2t-1), & 1/2 \le t \le 1 \end{cases}$$

   - J is continuous, since for all closed $V \subseteq Y$,

$$J^{-1}(V) = (J^{-1}(V) \cap (X \times [0,1/2])) \cup (J^{-1}(V) \cap (X \times [1/2,1])) = H^{-1}(V) \cup K^{-1}(V),$$

     and the closedness of $H^{-1}(V)$ and $K^{-1}(V)$ implies the closedness of $J^{-1}(V)$
   - Moreover, J has the property that $J(x,0) = H(x,0) = f(x)$, while $J(x,1) = K(x,1) = h(x)$.

■

R There are only one equivalence class in example (8.9). Actually, for given space X and Y, if any two continuous mapping are homotopic, then we imply there is only one equivalence class.

9.3. Monday for MAT4002

Reviewing.

1. Homotopy: we denote the homotopic function pair as $f \simeq g$.
2. If $Y \subseteq \mathbb{R}^n$ is convex, then the set of continuous functions $f : X \to Y$ form a single equivalence class, i.e., $\{\text{continuous functions } f : X \to Y\}/\sim$ has only one element

9.3.1. Remarks on Homotopy

Proposition 9.4 Consider four continous mappings

$$W \xrightarrow{f} X, \quad X \xrightarrow{g} Y, \quad X \xrightarrow{h} Y, \quad Y \xrightarrow{k} Z.$$

If $g \simeq h$, then

$$g \circ f \simeq h \circ f, \quad k \circ g \simeq k \circ h$$

Proof. Suppose there exists the homotopy $H : g \simeq h$, then $k \circ H : X \times I \to Z$ gives the momotopy between $k \circ g$ and $k \circ h$.

Simiarly, $H \circ (f \times \mathrm{id}_I) : W \times I \to Y$ gives the homotopy $g \circ f \simeq h \circ f$. ∎

Definition 9.4 [Homotopy Equivalence] Two topological spaces X and Y are **homotopy equivalent** if there are continuous maps $f : X \to Y$, and $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_{X \to X}$$
$$f \circ g \simeq \mathrm{id}_{Y \to Y},$$

which is denoted as $X \simeq Y$.

R

1. If $X \cong Y$ are homeomorphic, then they are homotopic equivalent.
2. The homotopy equivalence $X \simeq Y$ gives a bijection between $\{\phi : \text{continuous } W \to X\}/\sim$ and $\{\phi : \text{continuous } W \to Y\}/\sim$, for any given topological space W.

Proof. Since $X \simeq Y$, we can find $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$. We construct a mapping

$$\begin{aligned} \phi : \quad & \{\phi : \text{continuous } W \to X\}/\sim \to \{\phi : \text{continuous } W \to Y\}/\sim \\ \text{with} \quad & [\phi] \mapsto [f \circ \phi] \end{aligned}$$

ϕ is well-defined since $\phi_1 \sim \phi_2$ implies $f \circ \phi_1 \sim f \circ \phi_2$

Also, we can construct a mapping

$$\begin{aligned} \beta : \quad & \{\phi : \text{continuous } W \to Y\}/\sim \to \{\phi : \text{continuous } W \to X\}/\sim \\ \text{with} \quad & [\psi] \mapsto [g \circ \phi] \end{aligned}$$

Similarly, β is well-defined.

Also, we can check that $\alpha \circ \beta = \text{id}$ and $\beta \circ \alpha = \text{id}$. For example,

$$\alpha \circ \beta[\psi] = [f \circ g \circ \psi] = [\psi],$$

where the last equality is because that $f \circ g \simeq \text{id}_Y$. ■

3. The homotopy equivalence $X \simeq Y$ forms an equivalence relation between topological spaces

Compared with homeomorphism, some properties are lost when consider the homotopy equivalence.

Definition 9.5 [Contractible] The topological space X is **contractible** if it is homotopy equivalent to any point $\{c\}$.

(R) In other words, there exists continuous mappings f, g such that

$$\{c\} \xrightarrow{f} X \xrightarrow{g} \{c\}, \; g \circ f \simeq \text{id}_{\{c\}}$$
$$X \xrightarrow{g} \{c\} \xrightarrow{f} X, \; f \circ g \simeq \text{id}_X$$

Note that $g \circ f \simeq \text{id}_{\{c\}}$ follows naturally; and since $X \cong X$, we can find f, g

such that $f \circ g = c_y$ for some $y \in X$, where $c_y : X \to X$ is a constant function $c_y(x) = y, \forall x \in X$. Therefore, to check X is contractible, it suffices to check $c_y \simeq \mathrm{id}_X, \forall y \in X$.

Therefore, X is contractible if its identity map id_X is homotopic to any constant map $c_y, \forall y \in X$.

Proposition 9.5 The definition for contractible can be simplified further:

1. X is contractible if it is homotopy equivalent to some point $\{c\}$
2. X is contractible if the identity map id_X is homotopic to some constant map $c_y(x) = y$.

Proof. The only thing is to show that $c_y \simeq c_{y'}, \forall y, y' \in X$. By hw 3, X is path-connected, and therefore there exists continous $p(t)$ such that

$$p(0) = y, \quad p(1) = y'$$

Therefore, we construct the homotopy between c_y and $c_{y'}$ as follows:

$$H(x,t) = p(t).$$

■

Example 9.1 1. $X = \mathbb{R}^2$ is contractible:

It suffices to show that the mapping $f(\boldsymbol{x}) = \boldsymbol{x}, \forall \boldsymbol{x} \in \mathbb{R}^2$ is homotopic to the constant function $g(x) = (0,0), \forall x \in \mathbb{R}^2$, i.e., $g = c_{(0,0)}$.

Consider the continuous mapping $H(\boldsymbol{x},t) = tf(\boldsymbol{x})$, with

$$H(\boldsymbol{x},0) = c_{(0,0)}, \quad H(\boldsymbol{x},1) = \mathrm{id}_X$$

Therefore, $c_{(0,0)} \simeq \mathrm{id}_X$. Since $c_{(0,0)} \simeq c_{\boldsymbol{y}}, \forall \boldsymbol{y} \in \mathbb{R}^2$, we imply $c_{\boldsymbol{y}} \simeq \mathrm{id}_X$ for any $\boldsymbol{y} \in \mathbb{R}^2$. Therefore, X is contractible.

More generally, any convex $X \subseteq \mathbb{R}^n$ is contractible.

R S^1 is not contractible, and we will see it in 3 weeks' time. In particular, we are not able to construct the continuous mapping

$$H : S^1 \times [0,1] \to S^1$$

such that

$$H(e^{2\pi ix},0) = e^{2\pi ix}, \quad H(e^{2\pi ix},1) = e^{2\pi i(0)} = 1$$

How about the mapping $H(e^{2\pi ix},t) = e^{2\pi ixt}$? Unfortunately, it is not well-defined, since

$$H(e^{2\pi i(1)},t) = e^{2\pi it} = H(e^{2\pi i(0)},t) = 1$$

and the equality is not true for $t \neq 0,1$.

Definition 9.6 [Homotopy Retract] Let $A \subseteq X$ and $i : A \hookrightarrow X$ be an inclusion. We say A is a **homotopy retract** of X if there exists continuous mapping $r : X \to A$ such that

$$r \circ i : A \hookrightarrow X \xrightarrow{r} A = \mathrm{id}_A$$
$$i \circ r : X \xrightarrow{r} A \hookrightarrow X \simeq \mathrm{id}_X$$

In particualr, $A \simeq X$.

■ **Example 9.2** The 1-sphere S^1 is a homotopy retract of Mobius band M.

Let $M = [0,1]^2/\sim$ and $S^1 = [0,1]/\sim$. Define the inclusion i and r as:

$$i : \quad S^1 \hookrightarrow M$$
$$\text{with} \quad [x] \mapsto [(x, \tfrac{1}{2})]$$

$$
\begin{aligned}
r: \quad & M \to S^1 \\
\text{with} \quad & [(x,y)] \mapsto [x]
\end{aligned}
$$

As a result,

$$
r \circ i = \mathrm{id}_{S^1}, \qquad i \circ r([(x,y)]) = [(x,1/2)]
$$

It suffices to show $i \circ r \simeq \mathrm{id}_M$, where $\mathrm{id}_M([(x,y)]) = [(x,y)]$.

Construct the continous mapping $H : M \times I \to M$ with

$$
H([(x,y)],t) := [(x,(1-t)y + t/2)]
$$

To show the well-definedness of H, we need to check

$$
H([(0,y)],t) = H([(1,1-y)],t), \quad \forall y \in [0,1]
$$

It's clear that H gives a homotopy between $i \circ r$ and id_M, i.e., $i \circ r \simeq \mathrm{id}_M$ ■

■ **Example 9.3** The $n-1$-sphere S^{n-1} is a homotopy retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$:

We have the inclusion $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ and

$$
\begin{aligned}
r: \quad & \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1} \\
\text{with} \quad & x \mapsto \frac{x}{\|x\|}
\end{aligned}
$$

Therefore, $r \circ i = \mathrm{id}_{S^{n-1}}$ and $i \circ r(x) = \frac{x}{\|x\|}$.

It suffices to show that $i \circ r \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$. Consider the homotopy $H(x,t) = t\boldsymbol{x} + (1-t)\boldsymbol{x}/\|\boldsymbol{x}\|$ such that

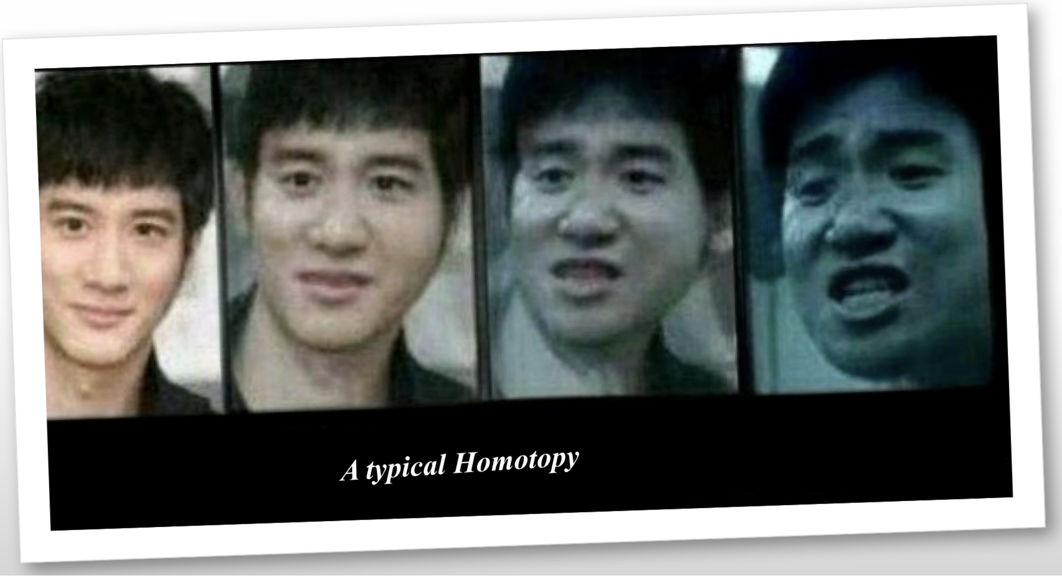
$$
H(\boldsymbol{x},0) = i \circ r(\boldsymbol{x}), \quad H(\boldsymbol{x},1) = \boldsymbol{x} = \mathrm{id}(\boldsymbol{x})
$$

To show the well-definedness of H, we need to check $H(x,t) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $t \in [0,1]$. ■

Definition 9.7 [Homotopic Relative] Let $A \subseteq X$ be topological spaces. We say $f,g : X \to Y$

are homotopic relative to A if there eixsts $H : X \times I \longrightarrow Y$ such that

$$\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases} \quad \text{and } H(a,t) = f(a) = g(a), \forall a \in A$$



9.6. Wednesday for MAT4002

9.6.1. Simplicial Approximation Theorem

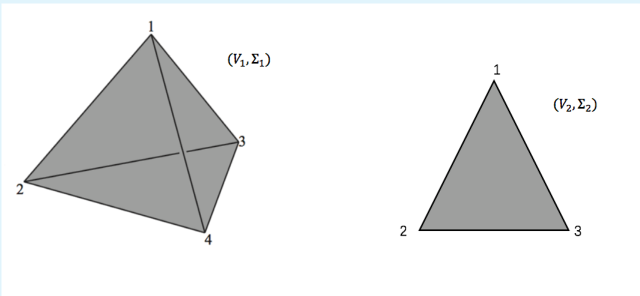
Aim: understand homotopy between simplicial complexes $f, g : |K| \to |L|$

Definition 9.12 [Simplicial Map] A **simplicial map** between $K_1 = (V_1, \Sigma_1)$ and $K_2 = (V_2, \Sigma_2)$ is a mapping $f : K_1 \to K_2$ such that

1. It maps vertexes to vertexes
2. It maps simplicies to simplicies, i.e.,

$$f(\sigma_1) \in \Sigma_2, \ \forall \sigma_1 \in \Sigma_1,$$

■ **Example 9.4** For instance, consider the simplicial complexes defined as follows:



In particular, $\{1,2,3,4\} \notin \Sigma_1$ and $\{1,2,3\} \in \Sigma_2$.

In this case, we can define the simplicial map as:

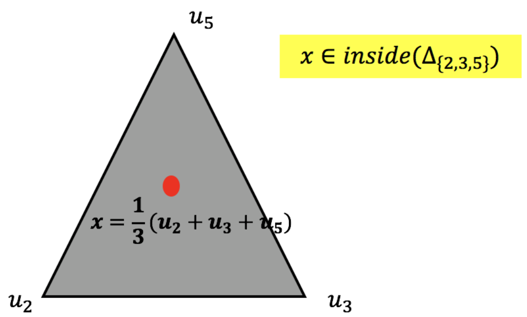
$$f(1) = 1, \quad f(2) = 2, \quad f(3) = 3, \quad f(4) = 3$$

In particular, $f(\{1,2,4\}) = \{1,2,3\} \in \Sigma_2$.

Now we want to define the simplicial map between the topological realizations. There are several observations:

Key Observations.

1. We have seen that each $|K| \subseteq \mathbb{R}^m$ for some m. In particular, $m = \#V - 1$.
2. Each point $x \in |K|$ lies uniquely on an inside of some Δ_{σ},, where $\sigma \in \Sigma$.
3. Suppose that the vertices of K_1 are $V_1 = \{u_1, \ldots, u_n\} \subseteq \mathbb{R}^m$. Then every $\boldsymbol{x} \in K_1$ can be uniquely written as
$$\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i U_{\sigma_i}$$
with $\alpha_i > 0, \sum \alpha_i = 1$ and $\sigma = \{U_{\sigma_1}, \ldots, U_{\sigma_k}\}$ is the unqiue simplex where $x \in$ inside(Δ_σ).



4. Our simplicial map f maps V_1 to $V_2 = \{w_1, \ldots, w_p\} \subseteq \mathbb{R}^m$, so for each i, we have $f(\boldsymbol{u}_i) = \boldsymbol{w}_j$ for some $j \in \{1, \ldots, p\}$.

Definition 9.13 [Mapping induced from Simplicial Mapping] The simplicial map $f : K_1 \to K_2$ induces a mapping $|f| : |K_1| \to |K_2|$ between the topological realizations such that

1. It maps vertexes to vertexes, i.e., $|f|(v_1) = f(v_1), \forall v_1 \in V(K_1)$.
2. it is affine, i.e.,
$$|f|\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} \alpha_i |f|(v_i)$$

R $|f| : |K_1| \to |K_2|$ is continuous.

Motivation. Suppose we are given a continuous map $|g| : |K| \to |L|$, we want to approximate $|g|$ by $|f|$, such that $f : K \to L$ is a simplicial map. In this case, f is an easier object to study compared with $|g|$.

We hope to find a mapping f such that $|f| \simeq |g|$. However, we cannot achieve this goal unless we subdivide K into smaller pieces:

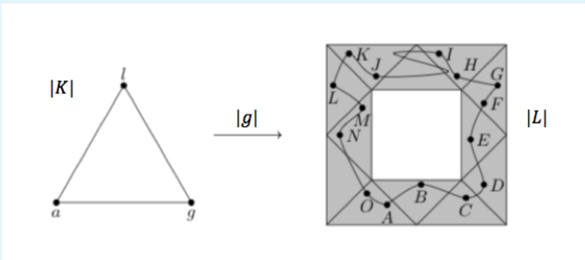
Definition 9.14 [Subdivision] Let K be a simplicial complex. A simplicial complex K' is called a **subdivision** of K if

1. Each simplex of K' is contained in a simplex of K
2. Each simplex of K equals the union of finitely many simplices of K'

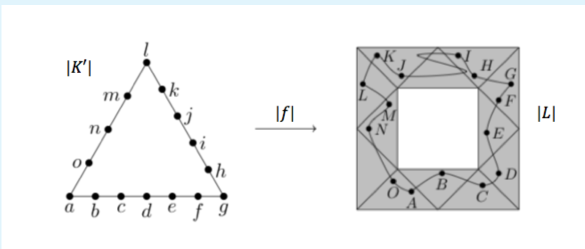
As a result, we can form an homeomorphism $h : |K'| \to |K|$ such that for each $\sigma' \in \Sigma_{K'}$, there exists $\sigma \in \Sigma_K$ satisfying

$$f(\Delta_{\sigma'}) \in \Delta_\sigma$$

■ **Example 9.5** Consider the mapping $|g| : |K| \to |L|$ given in the figure below:



Here we denote $|g|(a)$ by A and similarly for the other vertices. It's clear that we can not form a homeomorphism from $|K|$ to $|L|$. One remedy is to subdivide K into smaller pieces as follows:



In this case, it is clear that $|f| : |K'| \to |L|$ is a homeomorphism.

■ **Example 9.6** [Barycentric Subdivision] One typical subdivision is the Barycentric Subdivision:

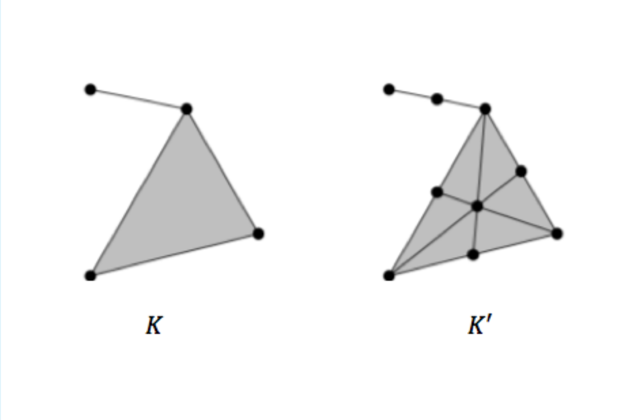


Figure 9.3: Right: the subdivision of K

■

R Suppose we have a matric on $|K|$. By subdivision, we can consider $|K'|$ such that for any $\sigma' \in \Sigma_{K'}$, any two points in $\Delta_{\sigma'}$ has a smaller distance.

The following result gives a criterion for the existence of a simplicial approximation for a mapping between topological realizations. For this we recall the notion of star. For a given simplicial complex K, define the star at a vertex v by

$$\text{star}(v) = \bigcup_{v\in\sigma} \sigma^\circ.$$

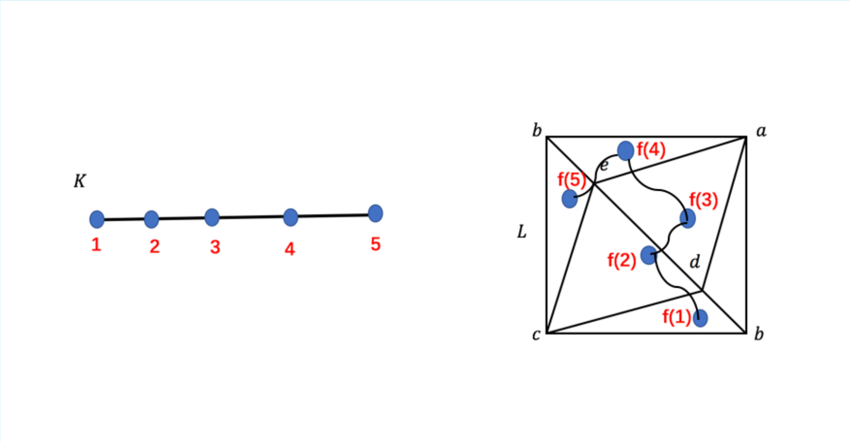
Proposition 9.11 Let $f : |K| \to |L|$ be a continuous mapping. Suppose that for each $v \in V_K$, there exists $g(v) \in V_L$ such that

$$f(\text{st}_K(v)) \subseteq \text{st}_L(g(v)),$$

then the mapping $g : V_K \to V_K$ gives $|g| \simeq f$.

In particular, g is called the **simplicial approximation** to f.

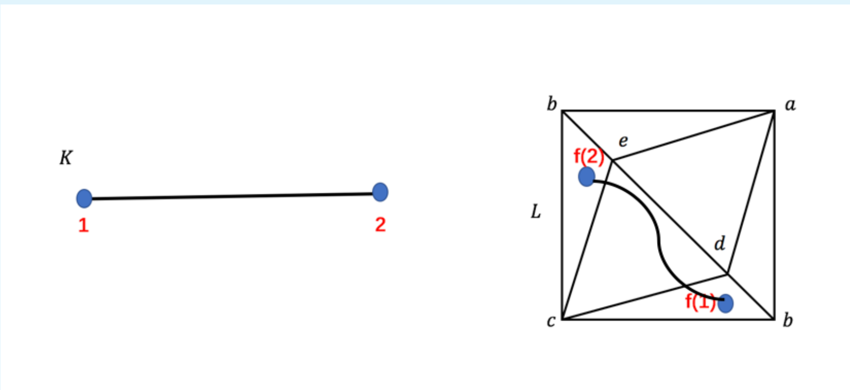
■ **Example 9.7** 1. First, we give an example of mapping f such that the assumption in proposition (9.11) is satisfied and therefore an simplicial approximation exists:



We could define the simplicial approximation g with

$$g(1) = b, g(2) = e, g(3) = e, g(4) = d, g(5) = d \text{ or } c$$

2. In the example below, the hypothesis of proposition (9.11) is not satisfied, so we cannot apply this proposition to construct a simplicial map.



■

Theorem 9.4 — **Simplicial Approximation.** Let K, L be simplicial complexes with V_K finite, and $f : |K| \to |L|$ be continuous. Then there eixsts a subdivision $|K'|$ of $|K|$ and a simplicial map g such that $|g| \simeq f$.

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Proposition 10.6 — Simplicial Approximation Proposition. Let K and L be two simplifical complexes, and $f : |K| \to |L|$ be a continuous mapping. If there exists a simplicial mapping $g : K \to L$ such that $f(\text{st}_K(\boldsymbol{v})) \subseteq \text{st}_L(g(\boldsymbol{v})), \forall \boldsymbol{v} \in V(K)$, then

$$|g| \simeq f$$

Recall the definition

$$\text{st}_K(\boldsymbol{v}) = \bigcup \{\text{inside}(\sigma) : \sigma \text{ is a simplex of } |K| \text{ and } x \in \sigma\}$$

Proof.

- We first show a statement: Suppose that $\sigma = \{v_0, \dots, v_n\} \in \Sigma(K)$, and $x \in \text{inside}(\sigma) \subseteq |K|$. If $f(x) \in |L|$ lies in the inside of the (unique) simplex $\tau \in \Sigma_L$, (i.e., $f(x)$ can uniquely be expressed as $\sum_{u_i \in \tau} \beta_i u_i$, such that $\beta_i > 0, \forall i$ and $\sum_i \beta_i = 1$) then $g(v_0), \dots, g(v_n)$ are vertices of τ.

  By definition of inside(σ), $x = \sum_{i=0}^n \alpha_i v_i$ with $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. Therefore, $x \in \text{st}_K(v_i)$ for $i = 1, \dots, n$, where

  $$\text{st}_K(v_i) := \left\{ av_i + \sum_{j=1}^m b_j w_j \mid a > 0, b_j > 0, a + \sum_{j=1}^m b_j = 1, \{v_i, w_1, \dots, w_m\} \in \Sigma_K \right\}.$$

  Therefore, $f(x) \in \text{int}(\text{st}_K(v_i)) \subseteq \text{st}_L(g(v_i))$, which follows that

  $$f(x) = ag(v_i) + \sum_{j=1}^m b_j u_j, \text{ where } a > 0, b_j > 0, a + \sum_{j=1}^m b_j = 1, \ \{g(v_i), u_1, \dots, u_m\} \in \Sigma_L$$

  Comparing the above formula with our hypothesis on $f(x)$, $g(v_i)$ is a vertex of the simplex τ, $i = 1, \dots, n$. Moreover, $\{g(v_0), \dots, g(v_n)\}$ is a subset of τ, which is a face of τ, and therefore $\{g(v_0), \dots, g(v_n)\} \in \Sigma_L$.

- Therefore, the mapping $g : K \to L$ maps simplicies to simplicies, which is a simplicial mapping. We can construct a homotopy between f and $|g|$ as follows: Consider any $x \in |K|$, and let $\tau \in \Sigma_L$ be such that $f(x) \in \text{inside}(\tau)$. We write

$x = \sum_{i=0}^{n} \lambda_i v_i$ for some $\{v_0, \ldots, v_n\} \in \Sigma_K$ and $\lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1$. Applying our claim,

$$|g|(x) = \sum_{i=0}^{n} \lambda_i g(v_i),$$

where $g(v_0), \ldots, g(v_n)$ are all vertices of τ.

We can directly construct a homotopy between f and $|g|$. Before that, we need some reformulations. Since $f(x) \in \text{inside}(\tau)$, we let $f(x) = \sum_{i=0}^{m} \mu_i \tau_i$. Since $|g|(x) = \sum_{i=0}^{n} \lambda_i g(v_i) \in \text{inside}(\tau)$, we rewrite $|g|(x) = \sum_{i=0}^{m} \lambda_i' \tau_i$. (by adding some $\lambda_i' := 0$ if necessary) We define the map

$$\begin{aligned} H: &\quad |K| \times I \to |L| \\ \text{with} &\quad (x,t) \mapsto \textstyle\sum_{i=0}^{m} t\lambda_i' + (1-t)\mu_i \end{aligned}$$

which follows that $f \simeq |g|$.

■

Theorem 10.2 — Simplicial Approximation Theorem. Let K,L be simplicial complexes with V_K finite, and $f : |K| \to |L|$ be continuous. Then there exists a subdivison $|K'|$ of $|K|$ together with a simplicial map g such that $|g| \simeq f$.

Here the way for constructing subdivison $|K'|$ is as follows. There exists a constant $\delta > 0$. As long as the coarseness of K' is less than δ, our constructed subdivision satisfies the condition.

Proof. The sets $\{\text{st}_L(w) \mid w \in V(L)\}$ forms an open cover of $|L|$, which implies $\{f^{-1}(\text{st}_L(w))\}$ forms an open cover of $|K|$. By compactness, there exists a finite subcover of $|K|$, denoted as

$$|K| \subseteq \bigcup_{i=1}^{n} f^{-1}(\text{st}_L(w_i))$$

There exists a small number $\delta > 0$ such that for any $x, y \in |K|$ with $d(x,y) < \delta$, $x, y \in f^{-1}(\text{st}_L(w_i))$ for some i. Then we construct a simplicial subdivision $|K'|$ of $|K|$ with coarseness less than δ, i.e., $\forall x, y \in \text{st}_{K'}(v)$, $d(x,y) < \delta$.

Therefore, $\text{st}_{K'}(v) \subseteq f^{-1}(\text{st}_L(w_i))$ for any $v \in V(K;)$ and some $w_i \in V(L)$, i.e., $f(\text{st}_{K'}(v)) \subseteq$

$\mathrm{st}_L(w_i)$.

Setting $g(v) = w_i$ and applying proposition (10.6) gives the desired result. ■

10.3.1. Group Presentations

Group is a highlight of our course, which interwises topology and algebra. I assume that most students have learnt abstract algebra course MAT3004, and encourage those without this knowledge to read the notes for group posted on blackboard.

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10.6.1. Reviewing On Groups

Example 10.4 Let D_{2n} be the regular polygon P with $2n$ sides in $\mathbb{R}^2$, centered at the origin. It's clear that D_{2n} is **invariant** with $2n$ rotations, or with 2 reflections. Let a denote the rotation of D_{2n} clockwise by degree π/n, and b denote the reflection over lines through the origin.

As a result, $\{e, a, a^2, \dots, a^{n-1}\}$ forms a group; and $\{e, b\}$ forms a group.

Therefore, all elements of D_g can be obtained by $a^i b^j, 0 \le i \le 3, 0 \le j \le 1$.

Any finite operations of rotation (the rotation degree is a multiple of π/n) and reflection can be represented as $a^i b^j$.

Geometrically, we can check that $ba = a^{n-1}b$.

Definition 10.7 [Product Group] Let G, H be two groups. The **product group** $(G \times H, *)$ is defined as

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$
$$\text{with} \quad (g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

For example, $(\mathbb{R} \times \mathbb{R}, +) = \{(x, y) \mid x, y \in \mathbb{R}\}$ coincides with the usual $\mathbb{R}^2$, where

$$(x, y) * (x', y') = (x + x', y + y')$$

Definition 10.8 A map between two groups $\phi : G \to H$ is a **homomorphism** if

$$\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$$

In other words, a homomorphism is a map preserving multiplications of groups.

R Follow the similar idea as in MAT3040 knowledge, if $\phi : G \to H$ is a homomorphism, then $\phi(e_G) = e_H$.

■ **Example 10.5** Let $G = (\mathbb{R}, +, 0)$, and $H = \{H_2, *, I_2\}$, with H_2 of the form

$$H_2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$$

Define a mapping

$$\begin{aligned} \phi : \quad & G \to H \\ \text{with} \quad & x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Then ϕ is a homorphism:

$$\begin{aligned} \phi(x *_{\mathbb{R}} y) &= \phi(x+y) \\ &= \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \\ &= \phi(x) *_{H_2} \phi(y) \end{aligned}$$

■

Definition 10.9 [Isomorphism] A homomorphism $\phi : G \to H$ is an isomorphism if ϕ is bijective. The isomorphism between G and H is denoted as $G \cong H$. ■

Actually, a group can be represented as a Cayley Table:

$$G = \begin{array}{c|cccc} \circ & g_1 & g_2 & \cdots & g_n \\ \hline g_1 & g_1 \circ g_1 & g_1 \circ g_2 & \cdots & g_1 \circ g_n \\ g_2 & g_2 \circ g_1 & g_2 \circ g_2 & \cdots & g_2 \circ g_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_n & g_n \circ g_1 & g_n \circ g_2 & \cdots & g_n \circ g_n \end{array}, \qquad H = \begin{array}{c|cccc} \circ & h_1 & h_2 & \cdots & h_n \\ \hline h_1 & h_1 \circ h_1 & h_1 \circ h_2 & \cdots & h_1 \circ h_n \\ h_2 & h_2 \circ h_1 & h_2 \circ h_2 & \cdots & h_2 \circ h_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_n & h_n \circ h_1 & h_n \circ h_2 & \cdots & h_n \circ h_n \end{array}$$

The groups $G \cong H$ if and only if we can find a bijective $\phi : G \to H$ such that, the Cayley

Table of $(H, \circ)$ can be generated from the Cayley Table of $(G, \circ)$ by replacing each entry of G with its image under ϕ.

10.6.2. Free Groups

Definition 10.10
- Let S be a (finite) set, which is considered as an "alphabet".
- Define another set $S^{-1} := \{x^{-1} \in x \in S\}$. We insist that $S \cap S^{-1} = \emptyset$.
- A **word** in S is a finite sequence $w = w_1 \cdots w_m$, where $m \in \mathbb{N}^+ \cup \{0\}$, and each $w_i = \in \cup S^{-1}$. In particular, when $m = 0$, we view w as the empty sequence, denoted as $\emptyset$.
- The **concatenation** of two words $x_1 \cdots x_m$ and $y_1 \cdots y_n$ is the word $x_1 \cdots x_m y_1 \cdots y_n$
- Two words w, w' are **equivalent**, denoted as $w \sim w'$, if there are words $w_1, \ldots, w_n$ and $w = w_1, w' = w_n$ such that

$$w_i = \cdots y_1 x x^{-1} y_2 \cdots, \qquad w_{i+1} = \cdots y_1 y_2 \cdots$$

or

$$w_i = \cdots y_1 y_2 \cdots, \qquad w_{i+1} = \cdots y_1 x x^{-1} y_2 \cdots$$

for some $x \in S \cup S^{-1}$.

■ **Example 10.6** For example, $S = \{a, b\}$ and $S^{-1} = \{a^{-1} b^{-1}\}$ and

$$w = aabab^{-1}b^{-1}a^{-1}abaabb^{-1}a$$
$$w' = aabab^{-1}b^{-1}a^{-1}abaaa$$

Here w and w' differs by bb^{-1}. Therefore, $w \sim w'$, and w is said to be a elementary expansion of w'.

R We insist that $(s^{-1})^{-1} = s, \forall s^{-1} \in S^{-1}$, since otherwise for $x = s^{-1} \in S^{-1}$, we cannot define $(s^{-1})^{-1}$.

Moreover, for

$$w = aabab^{-1}b^{-1}a^{-1}abaabb^{-1}a$$

$$w'' = aabab^{-1}b^{-1}baabb^{-1}a,$$

w and w'' differs by $a^{-1}a$, i.e., $a^{-1}(a^{-1})^{-1}$, and therefore $w \sim w''$.

Definition 10.11 [Free Group] The **free group** $F(S)$ is defined to be the equivalence class of words, i.e.,

$$[w] := \{w' \text{ is a word in } S \mid w \sim w'\} \in F(S)$$

R $F(S)$ is indeed a group:

- $[w] * [w'] = [ww']$ (concatenation) check $w_1 \sim w_2, u_1 \sim u_2$ implies $w_1u_1 \sim w_2u_2$
- Identity element: $e = [\emptyset]$
- Inverse element: $[x_1 \cdots x_n]^{-1} = [x_n^{-1} \cdots x_1^{-1}]$

■ **Example 10.7** Let $S = \{a\}$ and $S^{-1} = \{a^{-1}\}$. Any word w has the form

$$w = a \cdots aa^{-1} \cdots a^{-1}a \cdots aa^{-1} \cdots a^{-1} \cdots$$

In shorthand, we denote w as $w = \cdots a^p(a^{-1})^q a^r (a^{-1})^s \cdots$, and

$$\begin{aligned}[w] = [\cdots a^p(a^{-1})^q a^r (a^{-1})^s \cdots] &= [\cdots a^{p-1}(a^{-1})^{q-1} a^r (a^{-1})^s \cdots] \\ &= [\cdots a^{p-1}(a^{-1})^{q-2} a^{r-1} (a^{-1})^s \cdots],\end{aligned}$$

e.g., we can always eliminate the adjacent terms a and a^{-1} up to equivalence class. Therefore, $F(S) = \{\cdots, [a^{-2}], [a^{-1}], [\emptyset], [a], [a^2], \cdots\}$.

It's clear that $F(S) \cong \mathbb{Z}$, where the isomorphism $\phi : \mathbb{Z} \to F(S)$ is $\phi(n) = [a^n]$.

Example 10.8 Let $S = \{a, b\}$ and $S^{-1} = \{a^{-1}, b^{-1}\}$. In this case, $[ab] \neq [ba]$, and $[ab^{-1}a^2b^2a^{-2}b]$ cannot be reduced further.

Since S is not an abelian group in such case, we imply $F(S) \not\cong \mathbb{Z} \times \mathbb{Z}$.

10.6.3. Relations on Free Groups

Definition 10.12 [Group With Relations] Let S be a set. A **group with relations** is written as

$$G = \langle S \mid R(S) \rangle$$

where

- $R(S)$ consists of elements in $F(S)$
- Every element in G can be written as the form $[w] \in F(S)$, and we insist that $[w] = [w']$ in G if
  - w and w' differ by some $xx^{-1}, x \in S \cup S^{-1}$, or
  - w and w' differ by some element $z \in R(S)$, or its inverse.

Example 10.9 Let $G = \langle a, b \mid a^2, b^2, abab^{-1}a^{-1}b^{-1} \rangle$, we want to enumerate all possible elements in G. Obseve that

$$[b^{-1}] = [b^{-1}b^2] = [b], \quad \text{similarly } [a^{-1}] = [a]$$

$$[bab] = [abab^{-1}a^{-1}b^{-1}bab] = [abab^{-1}b] = [aba]$$

As a result,

- $[a^{-n}] = [a^n]$ and $[b^{-n}] = [b^n]$

- $[a^{2n+1}] = [a], [b^{2n+1}] = [b], [a^{2n}] = [\emptyset], [b^{2n}] = [\emptyset]$

- For another type of element of G, it must be of the form $[\cdot abababab\cdots]$.

  Each aba can be changed into bab, and finally it will be reduced into the form $[ab]$.

Therefore, the elements in G are

$$[\emptyset], [a], [b], [ab], [ba], [aba]$$

In fact, $G \cong S_3$. ■

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Reviewing. Consider the group with presentation $\langle S \mid R(S)\rangle$.

1. The elements in S are generators that have studied in abstract algebra
2. The "relations" of this group are given by the equalities on hte right-hand side, e.g., the dihedral group is defined as

$$\langle a, b \mid a^n = e, b^2 = e, bab = a^{-1}\rangle$$

Sometimes we also simplify the equality $\times = e$ as $\times$, e.g., the dihedral group can be re-written as

$$\langle a, b \mid a^n, b^2, bab = a^{-1}\rangle$$

Example 11.4 Consider

$$G =< a, b \mid a^2, b^2, abab^{-1}a^{-1}b^{-1} >:=< a, b \mid a^2, b^2, aba = bab >= \{e, a, b, ab, ba, aba\}$$

It's isomorphic to S^3, and the shape of S^3 is illustrated in Fig.(11.1)

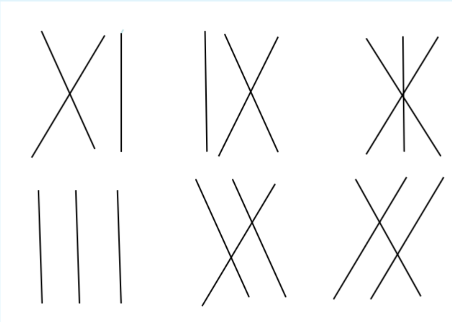


Figure 11.1: Illustration of group S^3

More precisely, the isomorphism is given by:

$$\begin{aligned} \phi :\quad & S_3 \to G \\ \text{with}\quad & X| \mapsto a, \quad |X \mapsto b \end{aligned}$$

■ **Example 11.5** Consider $G_2 =< a, b \mid ab = ba >$ and any word, which can be expressed as $\cdots a^s b^t a^u b^v \cdots$

- If $s \in \mathbb{N}$, we write $a^s := \underbrace{a \cdots a}_{s \text{ times}}$
- If $s \in -\mathbb{N}$, we write $a^s := \underbrace{(a^{-1}) \cdots (a^{-1})}_{-s \text{ times}}$
- For the word with the form $a \cdots b \cdots ba \cdots a$, we can always push a into the leftmost using the relation $ab = ba$
- For the word with the form $a \cdots ab \cdots ba^{-1}$, we can always push a^{-1} into the leftmost using the relation $ba^{-1} = a^{-1}b$.

Therefore, all elements in G_2 are of the form $a^p b^q, p, q \in \mathbb{Z}$, and we have the relation

$$(a^{p_1} b^{q_1})(a^{p_2} b^{q_2}) = a^{p_1+p_2} b^{q_1+q_2}.$$

Therefore, $G_2 \cong \mathbb{Z} \times \mathbb{Z}$, where the isomorphism is given by:

$$\begin{aligned} \phi: &\quad \mathbb{Z} \times \mathbb{Z} \to G_2 \\ \text{with} &\quad (p, q) \mapsto a^p b^q \end{aligned}$$

■

■ **Example 11.6**

$$G_3 = \langle a \mid a^5 \rangle = \{1, a, a^2, \dots, a^4\}$$

It's clear that $G_3 \cong \mathbb{Z}/5\mathbb{Z}$, where the isomorphism is given by:

$$\begin{aligned} \phi: &\quad \mathbb{Z}/5\mathbb{Z} \to G_3 \\ \text{with} &\quad m + 5\mathbb{Z} \mapsto a^m \end{aligned}$$

■

11.3.1. Cayley Graph for finitely presented groups

Graphs have strong connection with groups. Here we introduce a way of building graphs using groups, and the graphs are known as Cayley graphs. They describe many properties of the group in a topological way.

Definition 11.5 [Oriented Graph] An oriented graph T is specified by

1. A countable or finite set V, known as vertices
2. A countable or finite set E, known as edges
3. A function $\delta : E \to V \times V$ given by

$$\delta(e) = (\ell(e), \tau(e))$$

where $\ell(e)$ denotes the initial vertex and $\tau(e)$ denotes the terminal vertex.

For example, let

- $V = \{a,b,c\}$
- $E = \{e_1, e_2, e_3, e_4\}$
- $\delta(e_1) = (a,a), \delta(e_2) = (b,c), \delta(e_3) = (a,c), \delta(e_4) = (b,c)$

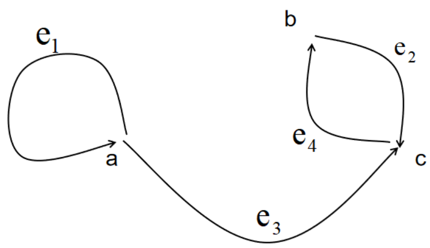


Figure 11.2: Illustration of example oriented graph

The resulted graph is plotted in Fig.(11.2)

Definition 11.6 [Cayley graph] Let $G = \langle S \mid R(S) \rangle$ with $|S| < \infty$. The **Cayley graph** associated to G is an oriented graph with

1. The vertex set G
2. The edge set $E := G \times S$
3. The function $\ell : E \to V \times V$ is given by:

$$\begin{aligned} \ell : \quad & G \times S \to G \times G \\ \text{with} \quad & (g,s) \mapsto (g, g \cdot s) \end{aligned}$$

In particular, we link two elements in G if they differ by a generator rightside.

■ **Example 11.7** 1. The Cayley graph for $G = \langle a \rangle$ ($\cong \mathbb{Z}$) is shown in Fig.(11.3):

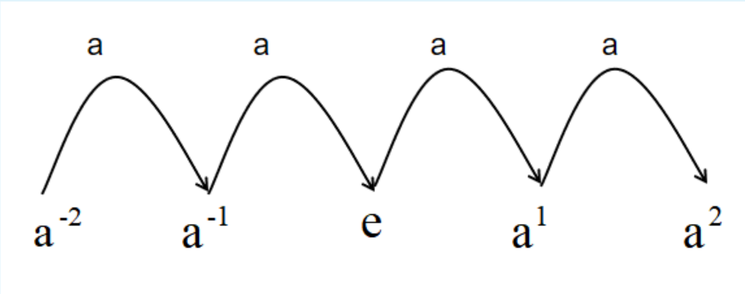


Figure 11.3: Illustration of Cayley Graph $\langle a \rangle$

2. The Cayley graph for $G = \langle a \mid a^3 \rangle$ is shown in Fig.(11.4):

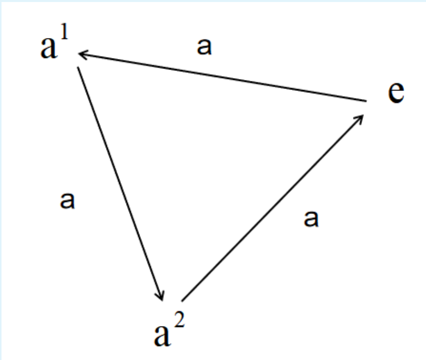


Figure 11.4: Illustration of Cayley Graph $\langle a \mid a^3 \rangle$

3. The Cayley graph for $G = \langle a, b \mid a^2, b^2, aba = bab \rangle$ is shown in Fig.(11.12):

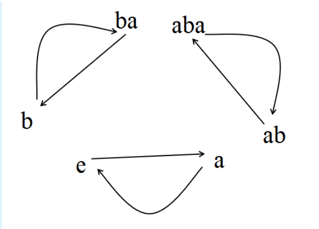


Figure 11.5: Illustration of Cayley Graph $\langle a,b \mid a^2,b^2,aba=bab\rangle$

4. The Cayley graph for $G=\langle a,b \mid ab=ba\rangle$ is shown in Fig.(11.6):

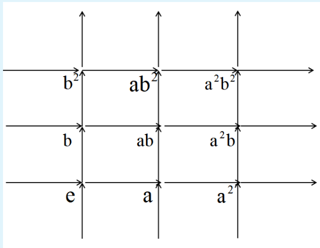


Figure 11.6: Illustration of Cayley Graph $\langle a,b \mid ab=ba\rangle$

5. The Cayley graph for $G=\langle a,b\rangle$ is shown in Fig.(11.7):

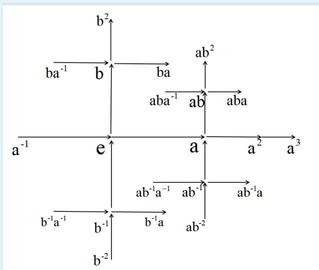


Figure 11.7: Illustration of Cayley Graph $\langle a,b \mid ab=ba\rangle$

R There could be different presentations $\langle S_1 \mid R(S_1)\rangle \cong \langle S_2 \mid R(S_2)\rangle$ of the same group.

11.3.2. Fundamental Group

Motivation. The fundamental group connects topology and algebra together, by labelling a group to each topological space, which is known as fundamental group.

Why do we need algebra in topology. Consider the S^2 (2-shpere) and $S^1 \times S^1$ (torus):

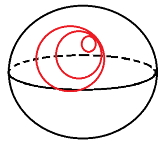


Figure 11.8: Any loop in the sphere can be contracted into a point

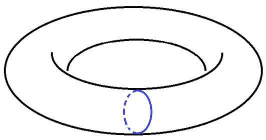


Figure 11.9: Some loops in the torus cannot be contracted into a point

As can be seen from Fig.(11.8) and Fig.(11.9), any "loop" on a sphere can be contracted to a point, while some "loop" on a torus cannot. We need the algebra to describe this phenomena formally.

Definition 11.7 [loop] Let X be a topological space. A **loop** on X is a constant map $\ell : [0,1] \to X$ such that $\ell(0) = \ell(1)$.

We say ℓ is based at $b \in X$ if $\ell(0) = \ell(1) = b$.

Definition 11.8 [composite loop] Suppose that $\boldsymbol{u}, \boldsymbol{v}$ are loops on X based at $b \in X$. The **composite loop** $u \cdot v$ is given by

$$u \cdot v = \begin{cases} u(2t), & \text{if } 0 \le t \le 1/2 \\ v(2t-1), & \text{if } 1/2 \le t \le 1 \end{cases}$$

Definition 11.9 [fundamental group] The **homotopy class of loops relative to** $\{0,1\}$ **based at** $b \in X$ forms a group. It is called the **fundamental group** of X based at b, denoted as $\pi_1(X,b)$.

More precisely, let

$$[\ell] = \{m \mid m \text{ is a loop based at } b \text{ that is homotopic to } \ell, \text{ relative to } \{0,1\}\},$$

and $\pi_1(X,b) = \{[\ell] \mid \ell \text{ are loops based at } b\}$. The operation in $\pi_1(X,b)$ is defined as:

$$[\ell] * [\ell'] := [\ell \cdot \ell'], \quad \forall [\ell],[\ell'] \in \pi_1(X,b).$$

R Two paths $\ell_1,\ell_2 : [0,1] \to X$ are homotopic relative to $\{0,1\}$ if we can find $H : [0,1]\times[0,1] \to X$ such that

$$H(t,0) = \ell_1(t), \quad H(t,1) = \ell_2(t)$$

and

$$H(0,s) = \ell_1(0) = \ell_2(0),\ \forall 0 \le s \le 1, \quad H(1,s) = \ell_1(1) = \ell_2(1),\ \forall 0 \le s \le 1$$

Counter example for homotopy but not relative to $\{0,1\}$:

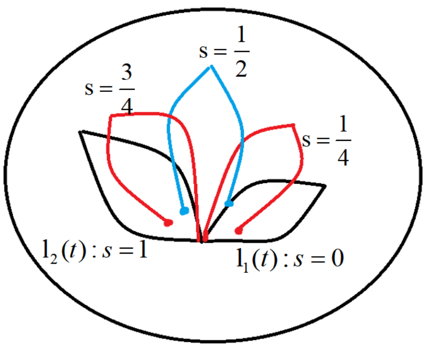


Figure 11.10: homotopy not relative to $\{0,1\}$

11.6. Wednesday for MAT4002

11.6.1. The fundamental group

Revewing. One example for Homotopy relative to $\{0,1\}$ is illustrated in Fig.(11.4)

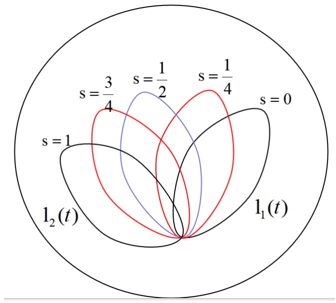


Figure 11.11: Example of homotopy relative to $\{0,1\}$

It's **essential** to study homotopy relative to $\{0,1\}$. For example, given a torus with a loop $\ell_1(t)$ and a base point b. We want to distinguish $\ell_1(t)$ and $\ell_2(t)$ as shown in Fig.(11.12):

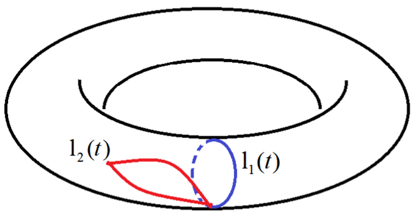


Figure 11.12: Two loops on a torus

Obviously there should be something different between $\ell_1(t)$ and $\ell_2(t)$. "Relative to$\{0,1\}$ is essential", sicne if we get rid of this condition, all loops are homotopic to the constant map $c_b(t) = b$. See the graphic illustration in Fig.(**??**):

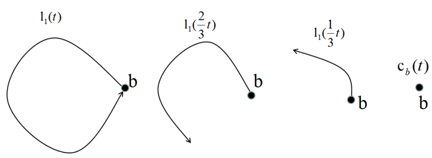


Figure 11.13: homotopy between any loop and constant map

In this case, $\ell \simeq c_b$ for any loop ℓ, there is only one trivial element $\{[c_b]\}$ in $\pi_1(X,b)$.

That's the reason why we define $\pi_1(X,b)$ as the collection of homotopy classes **relative to** $\{0,1\}$ based at b in X.

Proposition 11.13 Let $[\cdot]$ denote the homotopy class of loops relative to $\{0,1\}$ based at b, and define the operation

$$[\ell] * [\ell'] = [\ell \cdot \ell']$$

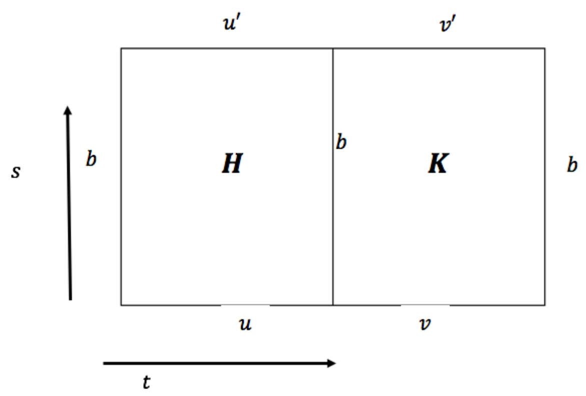
Then $(\pi_1(X,b), *)$ forms a group, where

$$\pi_1(X,b) := \{[\ell] \mid \ell : [0,1] \to X \text{ denotes loops based at } b\}$$

Proof. 1. Well-definedness: Suppose that $u \sim u'$ and $v \sim v'$, it suffices to show $u \cdot v \simeq u' \cdot v'$. Consider the given homotopies $H : u \simeq u'$, $K : v \simeq v'$. Construct a new homotopy $L : I \times I \to X$ by

$$L(t,s) = \begin{cases} H(2t,s), & 0 \le t \le 1/2 \\ K(2t-1,s), & 1/2 \le t \le 1 \end{cases}$$

The diagram below explains the ideas for constructing L. The plane denote the set $I \times I$, and the labels characterize the images of each point of $I \times I$ under L.

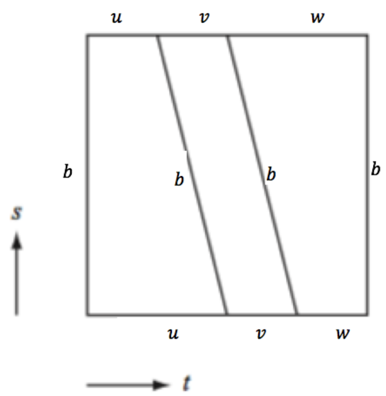


Therefore, $u \cdot v \simeq u' \cdot v'$.

2. Associate: $(u \cdot v) \cdot w \simeq u \cdot (v \cdot w)$

   Note that $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ are essentially different loops. Although they go with the same path, they are with different speeds. Generally speaking, the loop $(u \cdot v) \cdot w$ travels u, v using 1/4 seconds, and w in 1/2 seconds; but the loop $u \cdot (v \cdot w)$ travels u in 1/2 seconds, and then v, w in 1/4 seconds.

   We want to construct a homotopy that describes the loop changes from $u \cdot (v \cdot w)$ to $(u \cdot v) \cdot w$. A graphic illustration is given below:



   An explicit homotopy $H : I \times I \to X$ is given below:

$$H(t,s) = \begin{cases} u(4t/(2-s)), & 0 \le t \le 1/2 - 1/4s \\ v(4t - 2 + s), & 1/2 - 1/4s \le t \le 3/4 - 1/4s \\ w(4t - 3 + s/(1+s)), & 3/4 - 1/4s \le t \le 1 \end{cases}$$

   Therefore,

$$[u] * ([v] * [w]) = ([u] * [v]) * [w]$$

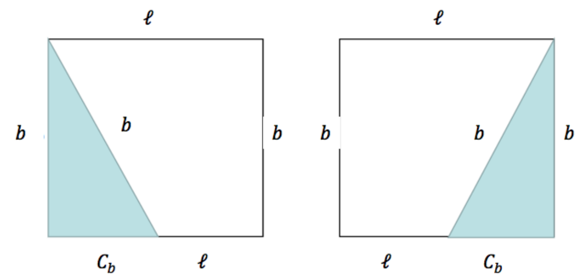
3. Intuitively, the identity should be the constant map, i.e., let $c_b : I \to X$ by $c_b(t) = b, \forall t$, and let $\ell = [c_b]$, it suffices to show

$$[c_b] * [\ell] = [\ell] * [c_b] = [\ell] \iff [c_b \cdot \ell] = [\ell \cdot c_b] = [\ell]$$

   Or equivalently,

$$c_b \cdot \ell \simeq \ell, \quad \ell \cdot c_b \simeq \ell$$

The graphic homotopy is shown below. (You should have been understood this diagram)



4. Inverse: the inverse of $[u]$, where u is a loop, should be $[u']$, where u' is the reverse of the traveling of u. Therefore, for all $u : I \longrightarrow X$ (loop based at b), define $u^{-1} : I \longrightarrow X$ by $u^{-1}(t) = u(1-t)$. Note that

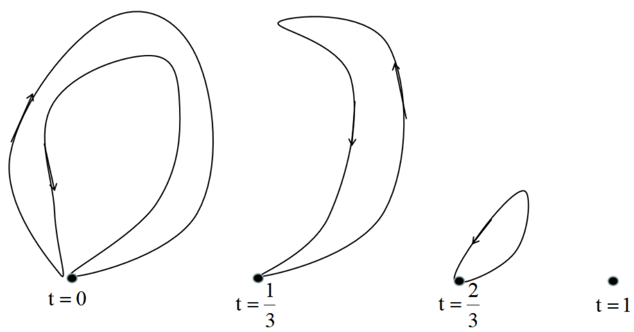
$$[u] * [u^{-1}] = [u \cdot u^{-1}], \quad e = [c_b]$$

It suffices to show $u \cdot u^{-1} \simeq c_b$ and $u^{-1} \cdot u \simeq c_b$:

The homotopy below gives $u \cdot u^{-1} \simeq c_b$, and the $u^{-1} \cdot u \simeq c_b$ follows similarly.

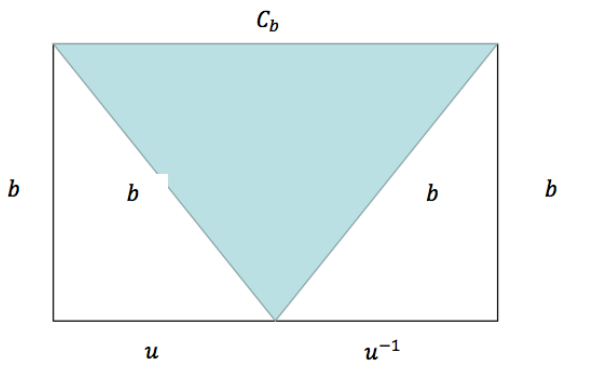
$$H(t,s) = \begin{cases} u(2t(1-s)), & 0 \leq t \leq 1/2 \\ u((2-2t)(1-s)), & 1/2 \leq t \leq 1 \end{cases}$$

The graphic illustration is given below:



■

R Note that the figure below does not define a homotopy from $u \cdot u^{-1}$ to c_b!



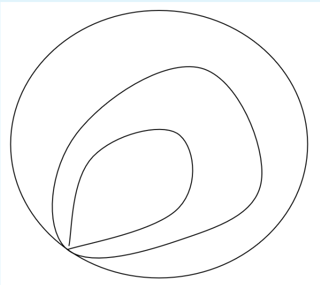
The reason is that for the upper part, as $s \to 1$, the time for traveling u and u^{-1} becomes very small, i.e., a particle has to pass u and u^{-1} in infinitely small time, which is not well-defined.

■ **Example 11.11** The reason why $\pi_1(\mathbb{R}^2, b) = \{e\}$ is trivial:

- For any $u : I \to \mathbb{R}^2$ with $u(0) = u(1) = b$, consider the homotopy

$$H(t,s) = (1-s)u(t) + sb.$$

Therefore, $u \simeq c_b$ for any loop u based at b. Check the diagram below for graphic illustration of this homotopy.



More generally, if $X \simeq \{x\}$ is contractible, then $\pi_1(X, b) = \{e\}$. The same argument cannot work for $(\mathbb{R}^2\{0\}, \boldsymbol{b})$, since the mapping $H : \mathbb{R}^2 \setminus \{0\} \times I \to \mathbb{R}^2 \setminus \{0\}$ with

$H(\boldsymbol{t}, s) = (1 - s)u(\boldsymbol{t}) + s\boldsymbol{b}$ is not well-defined. In particular, the value $H(s, t)$ may hit the origin $\mathbf{0}$.

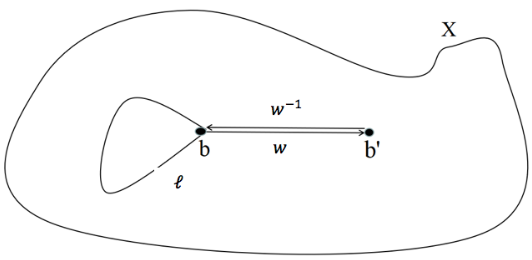
However, $\pi_1(S^1, 1)$ is non-trivial. We cannot deform the loop in S^1 into a constant loop. We will see that $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Proposition 11.14 If b, b' are path-connected in X, then $\pi_1(X, b) \cong \pi_1(X, b')$.

Proof. Let w be a path from b to b', and define

$$\begin{aligned} w_\# : \quad & \pi_1(X, b) \longrightarrow \pi_1(X, b') \\ \text{with} \quad & [\ell] \mapsto [w^{-1}\ell w] \end{aligned}$$

1. Well-definedness: Check that $\ell \simeq \ell'$ implies $w^{-1}\ell w \simeq w^{-1}\ell' w$. See the figure below for graphic illustration.



2. $w_\#$ is a homomorphism:

$$\begin{aligned} w_\#([\ell_1]) \cdot w_\#([\ell_2]) =& [w^{-1} \cdot \ell_1 w] \cdot [w^{-1} \cdot \ell_2 w] && (11.4a) \\ =& [w^{-1} \cdot \ell_1 \ell_2 w] && (11.4b) \\ =& w_\#([\ell_1 \ell_2]) && (11.4c) \end{aligned}$$

where (11.4b) is because that $w \cdot w^{-1} = c_b$.

3. And $w_\#$ is also injective. If loops ℓ_1, ℓ_2 are such that $w_\#(\ell_1) = w_\#(\ell_2)$, then

$$[w^{-1}\ell_1 w] = [w^{-1}\ell_2 w],$$

which follows that

$$[\ell_1] = [w][w^{-1}\ell_1 w][w^{-1}] = [w][w^{-1}\ell_2 w][w^{-1}] = [\ell_2] \tag{11.5}$$

4. Finally, $w_\#$ is surjective, because for any $u \in \pi_1(X, b')$, let $v = wuw^{-1}$, then v is based at b, so $[v] \in \pi_1(X, b)$, and $w_\#(v) = [u]$. Therefore $w_\#$ is surjective.

In conclusion, $w_\#$ is a group isomorphism between $\pi_1(X, b)$ and $\pi_1(X, b')$. ■

R In (11.5) we extended the meaning of $[\ell]$ to allow ℓ to be a path, and the equivalence class is defined by the relation "$\sim$": $\ell_1 \sim \ell_2$ iff they are homotopic relative to $\{0,1\}$. The multiplication rules are defined similarly.

12.3. Monday for MAT4002

Proposition 12.3 If b, b' are path connected in X, then

$$\pi_1(X,b) \cong \pi_1(X,b')$$

(R) Last lecture we have given the isomorphism

$$\begin{aligned} W_\# : \quad & \pi_1(X,b) \to \pi_1(X,b') \\ \text{with} \quad & [\ell] \mapsto [w^{-1} \cdot \ell \cdot w] \end{aligned}$$

where w denotes a path from b to b'. The inverse of $W_\#$ is given by:

$$\begin{aligned} W_\#^{-1} : \quad & \pi_1(X,b') \to \pi_1(X,b) \\ \text{with} \quad & [m] \mapsto [w \cdot m \cdot w^{-1}] \end{aligned}$$

Notation. For path connected space X, we will just write $\pi_1(X)$ instead of $\pi_1(X,x)$.

Proposition 12.4 Let (X,x) and (Y,y) be spaces with basepoints x and y, and $f : X \to Y$ be a continuous map with $f(x) = y$. Then every loop $\ell : I \to X$ based at x gives a loop $f \circ \ell : I \to Y$ based at y, i.e., the continous map f induces a homomorphism of groups

$$\begin{aligned} f_* : \quad & \pi_1(\pi,x) \to \pi_1(Y,y) \\ & [\ell] \mapsto [f \circ \ell] := f_*([\ell]) \end{aligned}$$

Moreover,

1. $(\mathrm{id}_{X \to X})_* = \mathrm{id}_{\pi_1(X,x) \to \pi_1(X,x)}$
2. $(g \circ f)_* = g_* \circ f_*$
3. If $f \simeq f'$ relative to $x \in X$, then $f_* = (f')_*$

Proof. • Well-definedness: Suppose that $\ell \simeq \ell'$, then $f \circ \ell \simeq f \circ \ell'$ by propositon (9.4). Therefore, $[f \circ \ell] = [f \circ \ell']$.

- Homomorphism: It's clear that

$$f \circ (\ell \circ \ell') = (f \circ \ell) \circ (f \circ \ell')$$

  Therefore, $f_*[\ell\ell'] = (f_*[\ell]) * (f_*[\ell'])$

The other three statements are obvious.

■

Proposition 12.5 Let X, Y be path-connected such that $X \simeq Y$ (i.e., there exists $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X, f \circ g \simeq \mathrm{id}_Y$). Then $\pi_1(X) \cong \pi_1(Y)$.

In particular, if X, Y are path-connected with $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$

Proof. Consider the mapping

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1)$$

It suffices to show that f_* and g_* are bijective. (The homomorphism follows from proposition (12.4))

- Wrong proof: $g \circ f \simeq \mathrm{id}_X$ implies $(g \circ f)_* = (\mathrm{id}_X)_*$ implies $g_* \circ f_* = \mathrm{id}_{\pi_1(X,x_0)}$.

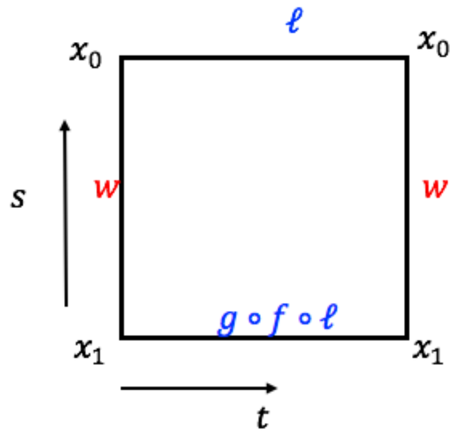
  Reason: note that $(g \circ f) \simeq \mathrm{id}_X$ is **not** relative to x_0.

Consider the homotopy $H : g \circ f \simeq \mathrm{id}_X$, where $H(x_0, s)$ is not necessarily a constant for $s \in I$. It follows that $H(x_0, 0) = x_1$ and $H(x_0, 1) = x_0$, i.e., $w(s) := H(x_0, s)$ defines a path from x_1 to x_0.

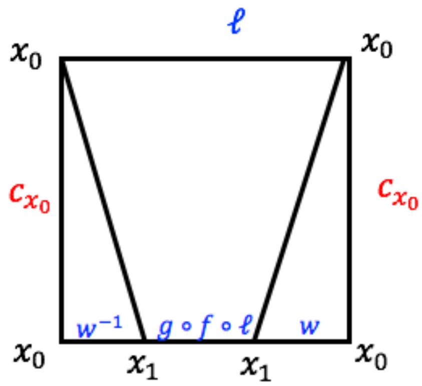
For any loop $\ell : I \to X$ based at x_0, consider the homotopy

$$\begin{aligned}
K = H \circ (\ell \times \mathrm{id}_I) : \quad & I \times I \to X \\
\text{where} \quad & K(t, s) = H((\ell(t), s)) \\
& K(t, 0) = H(\ell(t), 0) = g \circ f(\ell(t)) \\
& K(t, 1) = H(\ell(t), 1) = \ell(t) \\
& K(0, s) = w(s) = K(1, s)
\end{aligned}$$

The graphic plot of K is given in the figure below:



The homotopy between ℓ and $g \circ f \circ \ell$ motivates us to construct a homotopy between ℓ and $w^{-1} \circ g \circ f \circ \ell \circ w$ relative to $\{0,1\}$:



Therefore,

$$[\ell] = [w^{-1} g f \ell w] = W_{\#}([g f \ell]) = (W_{\#} \circ g_* \circ f_*)[\ell]$$

which follows that $W_{\#} \circ g_* \circ f_* = \mathrm{id}_{\pi_1(X,x_0)}$. Therfore, f_* is injective, g_* is surjective.

The similar argument gives

$$W_{\#} \circ f_* \circ g_* = \mathrm{id}_{\pi_1(Y,y_0)}$$

Therefore, f_* is surjective, g_* is injective. The bijectivity is shown.

∎

Definition 12.1 [Simply-Connected] A space X is **simply-connected** if X is path connected, and X has trivial fundamental group, i.e., $\pi_1(X) = \{e\}$ for some point $e \in X$.

▪ **Example 12.4** If X is contractible, then X is path-connected. By proposition (12.5), since $X \simeq \{e\}$, we imply

$$\pi_1(X) \cong \pi_1(\{e\}) = \{e\}.$$

Therefore, all contractible spaces (e.g., $\mathbb{R}^n$) are simply-connected.

However, not all simply-connected spaces are contractible, e.g., $\pi_1(S^2) \cong \{e\}$, but S^2 is not homotopy equivalent to a point.

12.3.1. Some basic results on $\pi_1(X, b)$

We will study $\pi_1(X, b)$ for some simplicial complexes.

Definition 12.2 [Edge Loop] Let $K = (V, \Sigma)$ be a simplicial complex.

1. An edge path $(v_0, \dots, v_n)$ is such that
   (a) $a_i \in V(K)$
   (b) For each i, $\{a_{i-1}, a_i\}$ spans a simplex of K
2. An edge loop is an edge path with $a_n = a_0$.
3. Let $\alpha = (v_0, \dots, v_n), \beta = (w_0, \dots, w_m)$ be two edge paths with $v_n = w_0$, then we define

$$\alpha \circ \beta = (v_0, \dots, v_n, w_1, \dots, w_m)$$

Definition 12.3 [Elementary Contraction/Expansion] Let α, β be two edge paths.

1. An elementary contraction of α is a new edge path obtained by performing one of the followings on α:

- Replacing $\cdots a_{i-1}a_i\cdots$ by $\cdots a_{i-1}\cdots$ provided that $a_{i-1}=a_i$
- Replacing $\cdots a_{i-1}a_ia_{i+1}\cdots$ by $\cdots a_{i-1}\cdots$ provided that $a_{i-1}=a_{i+1}$
- Replacing $\cdots a_{i-1}a_ia_{i+1}\cdots$ by $\cdots a_{i-1}a_{i+1}\cdots$ provided that $\{a_{i-1},a_i,a_{i+1}\}$ spans a 2-simplex of K.

2. An elementary expansion is the reverse of the elementary contraction.
3. Two edge paths α,β are equivalent if α and β differs by a finite sequence of elementary contractions or expansions.

12.6. Wednesday for MAT4002

Reviewing.

- Edge loop based at $b \in V$:
$$\alpha = (b, v_1, \cdots, v_n, b)$$

- Equivalence class of edge loops:
$$[\alpha] = \{\alpha' \mid \alpha' \sim \alpha, \alpha' \text{ is the edge loop based at } b\}$$

  Note that $\alpha' \sim \alpha$ if they differ from finitely many elementary contractions or expansions.

  For instance, let K in the figure below denote a triangle:

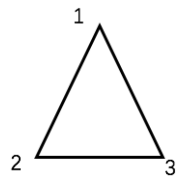


Figure 12.1: Triangle K

  Then the canonical form of any equivalence form $[\alpha]$ can be expressed as:
$$[\alpha] = [bcabc \cdots ab],$$
  where $a, b, c \in \{1, 2, 3\}$ are distinct.

12.6.1. Groups & Simplicial Complices

Proposition 12.7 The $E(K, b) = \{[\alpha] \mid \alpha \text{ is edge loop based at } b\}$ is a group, with the operation
$$[\alpha] * [\beta] = [\alpha \cdot \beta]$$

Proof. 1. Well-definedness of $*$:

$$\alpha \sim \alpha', \beta \sim \beta' \implies \alpha \cdot \beta \sim \alpha' \cdot \beta'$$

2. Associativity is clear.
3. The identity is $e := [b]$: for any edge loop $[\alpha] = [bv_1 \cdots b]$,

$$\begin{aligned}
[\alpha] * e &= [bv_1 \cdots v_n b] * [b] \\
&= [bv_1 \cdots v_n bb] \\
&= [bv_1 \cdots v_n b] = [\alpha].
\end{aligned}$$

   Also, $e * [\alpha] = [\alpha]$.
4. The inverse of any edge loop $[bv_1 \cdots v_n b]$ is $[bv_n \cdots v_1 b]$:

$$\begin{aligned}
[bv_1 \cdots v_n b]^{-1} * [bv_1 \cdots v_n b] &= [bv_n \cdots v_1 bbv_1 \cdots v_n b] \\
&= [bv_n \cdots v_1 bv_1 \cdots v_n b] \\
&= [bv_n \cdots v_2 v_1 v_2 \cdots v_n b] \\
&= \cdots \\
&= [b]
\end{aligned}$$

   Similarly, $[bv_1 \cdots v_n b] * [bv_1 \cdots v_n b]^{-1} = [b]$.

■

We will see that for K defined in Fig.(12.1), $E(K, 1) \cong \mathbb{Z}$, in the next class.

Theorem 12.5 $E(K, b) \cong \pi_1(|K|, b)$.

This is the most difficult theorem that we have faced so far. Let's recall the simplicial approximation proposition first:

R

Proposition 12.8 — Simplicial Approximation Proposition. Suppose that $f : |K| \to |L|$ be such that for all $v \in V(K)$, there exists $g(v) \in V(L)$ satisfying

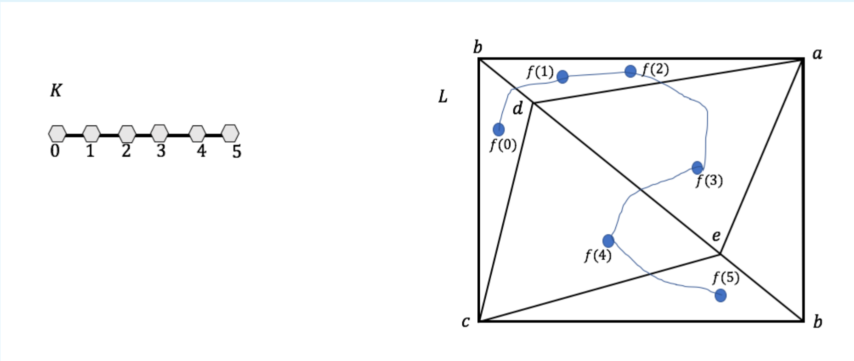
$$f(\mathrm{st}_K(v)) \subseteq \mathrm{st}_L(g(v)).$$

As a result,

1. the mapping
$$\begin{aligned} g: \quad & K \to L \\ \text{with} \quad & v \mapsto g(v) \end{aligned}$$
is a simplicial map, i.e., for all $\sigma_K \in \Sigma_K, g(\sigma_K) \in \Sigma_L$
2. Moreover, $|g| \simeq f$.

Furthermore, if $A \subseteq K$ is a simplicial subcomplex such that $f(|A|) \subseteq |B|$, where $B \subseteq L$ is a simplicial subcomplex, then we can choose g such that $g\mid_A : A \to B$ and the homotopy $|g| \simeq f$ sends $|A|$ to $|B|$.

■ **Example 12.8** Consider the simplicial complex K and L shown in the figure below:



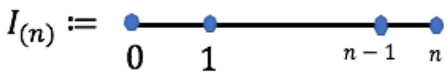
Let A_1 denote the subcomplex with $V(A_1) = \{0\}, \Sigma_{A_1} = \{\{0\}\}$, and A_2 denote the subcomplex wit $V(A_2) = \{1,2\}$ and $\Sigma_{A_2} = \{\{1,2\},\{1\},\{2\}\}$. Therefore,

$$f(|A_1|) \subseteq |\Delta_{\{b,c,d\}}|, \quad f(|A_2|) \subseteq |\Delta_{\{a,b,d\}}|,$$

There exists simplicial mapping g with

$$g(0)=b,\quad g(1)=b,\quad g(2)=d,\quad g(3)=e,\quad g(4)=c,\quad g(5)=c$$

Proof. 1. For each edge loop $\alpha=(v_0,\ldots,v_n)$ based at b, consider the simplicial complex



Together with the simplicial map

$$\begin{aligned} g_\alpha :\quad & I_{(n)} \longrightarrow K \\ \text{with}\quad & g_\alpha(i)=v_i \end{aligned}$$

Note that it is well-defined since $\{i,i+1\}\in\Sigma_{I_{(n)}}$, and $\{v_i,v_{i+1}\}\in\Sigma_K$.

Now construct the mapping

$$\begin{aligned} \theta :\quad & \{\text{edge loop based at } b\} \longrightarrow \pi_1(K,b) \\ \text{with}\quad & \alpha \mapsto [|g_\alpha|] \\ \text{where}\quad & |g_\alpha| : |I_{(n)}|(\cong [0,1]) \longrightarrow |K| \\ & |g_\alpha|(i/n)=v_i \end{aligned}$$

For example,

$$\alpha=(bdeabcb),\implies |g_\alpha|(0)=b,|g_\alpha|(1/6)=d,|g_\alpha|(2/6)=e,\cdots,|g_\alpha|(1)=b,$$

i.e., $|g_\alpha|$ is a loop based at b.

Therefore, $[|g_\alpha|]\in\pi_1(|K|,b)$.

2. Now, suppose $\alpha \sim \alpha'$ be two edge loops differ by an elemenary contraction, e.g.,

$$\alpha' = (bdebcb) \sim \alpha = (bdeabcb).$$

As a result, $|g_{\alpha'}| \simeq |g_\alpha|$ relative to $\{0,1\}$, i.e., $[|g_\alpha|] = [|g_{\alpha'}|]$.

Therefore, we have a well-defined map:

$$\begin{aligned} \tilde{\theta}: &\quad \{\text{edge loops based at } b\}/\sim \longrightarrow \pi_1(|K|, b) \\ \text{with} &\quad [\alpha] \mapsto [|g_\alpha|] \end{aligned}$$

Therefore, $\tilde{\theta}: E(K,b) \to \pi_1(|K|,b)$ is the desired map.

3. $\tilde{\theta}$ is a homomorphism: it suffices to show that

$$\tilde{\theta}([\alpha] * [\beta]) = \tilde{\theta}([\alpha])\tilde{\theta}([\beta]),$$

which suffices to show $[|g_{\alpha\cdot\beta}|] = [|g_\alpha||g_\beta|]$, i.e., $|g_{\alpha\cdot\beta}| \simeq |g_\alpha||g_\beta|$. Note that $|g_{\alpha\cdot\beta}|$ and $|g_\alpha||g_\beta|$ are the same path with different "speed", i.e., homotopy.

4. The mapping $\tilde{\theta}$ is surjective: Let $\ell : [0,1] \to |K|$ be a loop based at b. It suffices to find an edge loop α such that $[|g_\alpha|] = [\ell]$, i.e., $|g_\alpha| \simeq \ell$.

   (a) Applying the simplicial approximation theorem, there exist large n and $g : I_{(n)} \to K$ such that $|g| \simeq \ell$. Here we can choose $g : I_{(n)} \to K$ to be such that $g(\{0\}) = \{b\}$, $g(\{n\}) = \{b\}$, and $|g| \simeq \ell$ relative to $\{0,1\}$.

   (b) Take $\alpha = (g(0), g(1), \ldots, g(n))$ so that $g(0) = b = g(n)$, with $g_\alpha = g$. Therefore, $[|g_\alpha|] = [\ell]$, and hence $\tilde{\theta}$ is surjective.

■

13.3. Monday for MAT4002

13.3.1. Isomorphsim between Edge Loop Group and the Fundamental Group

Recall that

$$\pi_1(X,b) := \{[\ell] \mid \ell : [0,1] \to X \text{ denotes the loops based at } b\}$$

and

$$E(K,b) = \{[\alpha] \mid \alpha \text{ is an edge loop in } K \text{ based at } b\}$$

Now we show that the mapping defined below is injective:

$$\begin{aligned} \theta : \quad & E(K,b) \to \pi_1(|K|,b) \\ \text{with} \quad & [\alpha] \mapsto [|g_\alpha|] \end{aligned}$$

- Let $\alpha = (v_0,\dots,v_n)$ be an edge loop based at b such that $\theta([\alpha]) = e$, i.e., $|g_\alpha| \simeq c_b$. It suffices to show that $[\alpha]$ is the identity element of $E(K,b)$.
- Choose a homotopy $H : |g_\alpha| \simeq c_b$ such that $H : I \times I \to |K|$. The graphic illustration for H is shown in Fig. (13.8).

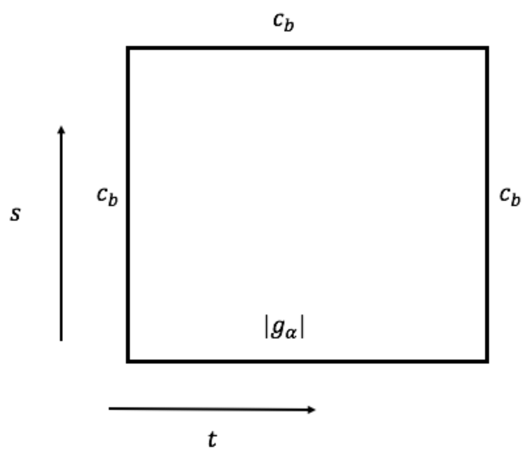


Figure 13.1: Graphic illustration for $H : I \times I \to |K|$

Now apply the simplicial approximation theorem, there exists a subdivision of $I \times I$, denoted as $(I \times I)_{(r)}$ (for sufficiently large r), shown in the Fig. (13.9)

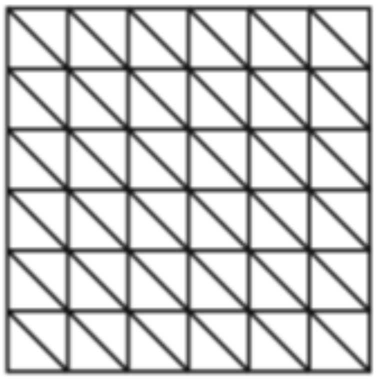


Figure 13.2: Graphic illustration for $(I \times I)_{(r)}$. In particular, divide $I \times I$ into r^2 congruent squares, and then further divide each of these squares along the diagonal to form $(I \times I)_{(r)}$.

such that $|(I \times I)_{(r)}| = I \times I$, and there exists the simplicial map

$$\begin{aligned} G: &\quad (I \times I)_{(r)} \rightarrow K \\ \text{such that} &\quad |G| \simeq H. \end{aligned}$$

Without loss of generality, assume r is a sufficiently large multiple of n.

The graphic illustration of $|G|$ is shown in Fig. (13.3):

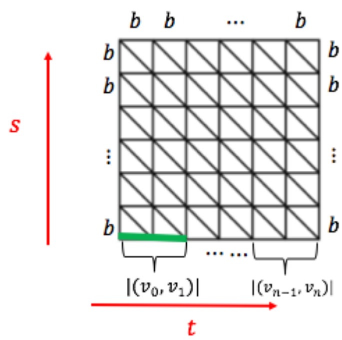


Figure 13.3: Graphic illustration for the mapping $|G|$.

In particular, $|G|$ maps $\{0,1\} \times I$ into $\{b\}$; $I \times \{1\}$ into $\{b\}$; $(i/n, 0)$ into $\{v_i\}, i =$

$0, \ldots, n$, and $[i/n, (i+1)/n]$ into $|(v_i, v_{i+1})|, i = 0, \ldots, n-1$.

- Consider the simplicial subcomplex of $(I \times I)_{(r)}$ shown in Fig. (13.4)

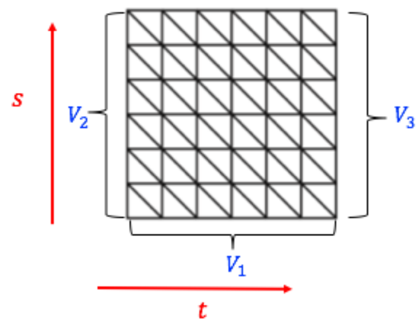


Figure 13.4: Graphic illustration for the simplicial subcomplex V_1, V_2, V_3.

For instance, V_1 has $(r+1)$ 0-simplicies and r 1-simplies. It follows that

$$H(|V_1|) = H(|V_2|) = H(|V_3|) = \{b\}.$$

By proposition (10.6), we can pick G be such that

$$G(V_1) = G(V_2) = G(V_3) = \{b\}.$$

Consider W_1 as the simplicial subcomplex of $(I \times I)_{(r)}$ given by the green line shown in Fig. (13.3), which follows that

$$H(|W_1|) = \{v_0, v_1\} \implies G(W_1) = \{v_0, v_1\}$$

Similarly,

$$H(|W_i|) = \{v_{i-1}, v_i\} \implies G(W_i) = \{v_{i-1}, v_i\}, \forall 1 \leq i \leq n.$$

As a result, $|G|(|V_1|) = \beta := (bv_0 \cdots v_0 v_1 \cdots v_1 \cdots v_n \cdots v_n b)$, and clearly,

$$\begin{aligned}\beta &\sim (bv_0 v_1 v_2 \cdots v_{n-1} v_n b)\\ &\sim (bv_1 v_2 \cdots v_{n-1} b) = \alpha\end{aligned}$$

- Now it suffices to show $\beta \simeq e$. This is true by the sequence of elementary contractions and expansions as shown in the Fig. (13.5).

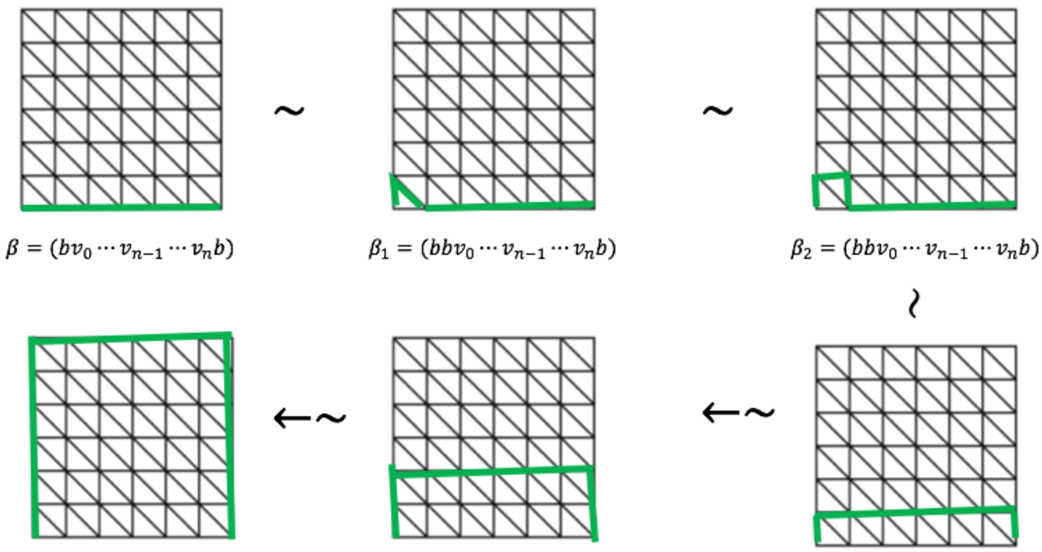


Figure 13.5: A sequence of elementary contractions and expansions to show that $\beta \sim (b \cdots b) = (b)$.

R The definition of $E(K,b)$ only involves n-simplicials for $n \leq 2$.

Proposition 13.4 For any simplicial complex K, consider the simplicial subcomplex $\text{Skel}^n(K) = (V_k, \Sigma_K^n)$, where Σ_K^n consists of $\sigma \in \Sigma_K$ with $|\sigma| \leq n+1$ (this is the n-skeleton of K). Then

$$\pi_1(|K|,b) \cong \pi_1(|\text{Skel}^2(K)|,b)$$

Proof. Since $E(K,b)$ only involves n-simplicials for $n \leq 2$, we imply $E(K,b) \cong E(\text{Skel}^2(K),b)$.

Moreover, $\pi_1(|K|,b) \cong E(K,b)$ and $\pi_1(|\text{Skel}^2(K)|,b) \cong E(\text{Skel}^2(K),b)$.

The proof is complete. ■

Corollary 13.2 For $n \geq 2$, $\pi_1(S^n)$ is a trivial fundamental group.

Proof. Consider the simplicial complex K with

$$V = \{1,2,\ldots,n+2\}, \quad \Sigma = \{\text{all proper subsets of } V\}$$

It's clear that $|K| \cong S^n$, and $\text{Skel}^2(K)$ has

- $V : \{1, \dots, n+2\}$
- Σ^2 : all subsets of V with less or equal to 3 elements.

For any edge loop a in $\pi_1(|\text{skel}^2(K)|)$, we have

$$\begin{aligned}
a &= (bv_0v_1v_2\cdots v_n) \\
&\sim (bv_1v_2\cdots v_{n-2}v_{n-1}b) \\
&\sim \cdots \\
&\sim (b)
\end{aligned}$$

Therefore, all edge loops α in $\pi_1(|\text{skel}^2(K)|)$ satisfies $[\alpha] = [(b)] = e.$, i.e.,

$$\pi_1(|\text{skel}^2(K)|) \cong \{e\},$$

which implies $\pi_1(|K|) \cong \pi_1(|\text{skel}^2(K)|) \cong \{e\}$. Since $|K| \cong S^n$, we imply

$$\pi_1(S^n) \cong \pi_1(|K|) \cong \{e\}.$$

■

R The Corollary (13.2) does not hold for S^1 since the constructed Σ^2 for S^1 does not contain $\{1,2,3\}$.

Theorem 13.4 $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. Construct the triangle K shown in Fig. (13.6), and it's clear that $|K| \cong S^1$.

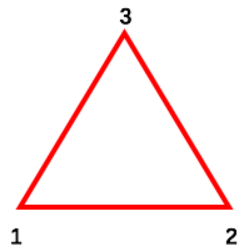


Figure 13.6: Triangle K such that $|K| \cong S^1$

It suffices to show $E(K,1) \cong \mathbb{Z}$. Define the orientation of $|K|$ as shown in Fig. (13.7).

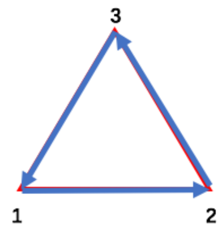


Figure 13.7: Orientation of $|K|$

Any edge loop α based at 1 is equivalent to the canonical form

$$\alpha \sim (1bc1bc\cdots 1bc1), \quad \text{where } bc = 32 \text{ or } 23.$$

We construct the isomorphism between $E(K,b)$ and $\mathbb{Z}$ directly:

$$\begin{aligned} \phi : \quad & E(K,b) \to \mathbb{Z} \\ \text{with} \quad & [\alpha] \mapsto \text{winding number of } \alpha \end{aligned}$$

where the winding number of α is the number of times it traverses $(2,3)$ in the forwards direction minus the number of times it traverses $(3,2)$ in the backwards direction.

The difficult part is to show the well-definedness of ϕ, which can be done by using canonical form of α. ■

13.6. Wednesday for MAT4002

13.6.1. Applications on the isomorphism of fundamental group

Theorem 13.6

$$\pi_1(S^1) \cong (\mathbb{Z}, +)$$

Proof. Define the orientation of $|K|$ as shown in Fig. (13.10).

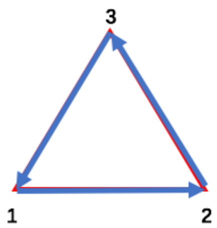


Figure 13.10: Orientation of $|K|$

Following the proof during last lecture, we construct

$$\begin{aligned} \phi : \quad & E(K,1) \to (\mathbb{Z}, +) \\ \text{with} \quad & [\alpha] \mapsto \text{winding number of } \alpha \end{aligned}$$

where the winding number of α is the

$$\text{number of 23 appearing in } \alpha - \text{number of 32 appearing in } \alpha.$$

Note that

1. The winding number is invariant under elementary contraction and elementary expansion.

2. In particular,

$$\text{winding number for } (1 \quad \underbrace{23\cdots123}_{\text{23 shows for } m \text{ times}} \quad 1) = m$$
$$\text{winding number for } (1 \quad \underbrace{32\cdots132}_{\text{32 shows for } n \text{ times}} \quad 1) = -n$$

3. For any given α, it is equivalent to a unique $(123123\cdots1231)$ or $(132\cdots1321)$, since otherwise α will have different winding numbers.

Therefore, (1) and (3) shows the well-definedness of ϕ. In particular, (1) shows that as $\alpha \sim \alpha'$, we have $\phi([\alpha]) = \phi([\alpha'])$; (2) shows that the winding number of α is an unique integer.

- Homomorphism: For given any two edge loops α, β based at 1, suppose that $[\alpha] = [(1bc1bc\cdots1bc1)]$ and $[\beta] = [(1pq1pq\cdots1pq1)]$, then

$$\phi([\alpha]\cdot[\beta]) = \phi([\alpha\cdot\beta]) = [(1bc1bc\cdots1bc11pq1pq\cdots1pq1)]$$

  Discuss the case for the sign of $\phi([\alpha])$ and $\phi([\beta])$ separately gives the desired result.

- Surjectivity: for a given $m \in \mathbb{Z}$, construct α such that $\phi([\alpha]) = m$ is easy.

- Injectivity: suppose that $\phi([\alpha]) = 0$, then by definition of ϕ, $[\alpha] = [(1)] = e$, which is the trivial element in $E(K,1)$.

Therefore, ϕ is an isomorphism. ■

R Actually, we can show that the loop based at 1 given by:

$$\begin{aligned} \ell \quad & I \to S^1 \\ \text{with} \quad & t \mapsto e^{2\pi i t} \end{aligned}$$

is a generator for $\pi_1(S^1, 1)$:

- $\phi([\ell]) = 1$, where $\phi : \pi_1(S^1, 1) \cong \mathbb{Z}$.
- The loop

$$\begin{aligned} \ell^m : \quad & I \to S^1 \quad m \in \mathbb{Z} \\ \text{with} \quad & \ell^m(t) = e^{2\pi i m t} \end{aligned}$$

  gives $\phi([\ell^m]) = m$

Corollary 13.4 [Fundamental Theorem of Algebra] All non-constant polynomials in $\mathbb{C}$ has at least one root in $\mathbb{C}$

Proof.

- Suppose on the contrary that

$$p(x) = a_n x^n + \cdots + a_1 x + a_0 \ a_n \neq 0$$

  has no roots, then p is a mapping from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$. It's clear that $\mathbb{C} \setminus \{0\} \simeq \{z \in \mathbb{C} \mid |z| = 1\}$, and therefore

$$\pi_1(\mathbb{C} \setminus \{0\}) = \pi_1(S^1) \cong \mathbb{Z}.$$

- The induced homomorphism p^* of p is given by:

$$\begin{aligned} p_* : \quad & \pi_1(\mathbb{C}) \to \pi_1(\mathbb{C} \setminus \{0\}) \\ \text{with} \quad & \{e\} \mapsto \mathbb{Z} \end{aligned}$$

  Note that $\pi_1(\mathbb{C})$ is trival as $\mathbb{C}$ is contractible. Also, $p_*(e) = 0$.

- Consider the inclusion from $C_r = \{z \in \mathbb{C} \mid |z| = r\}$ to $\mathbb{C}$:

$$\begin{aligned} i : \quad & C_r \to \mathbb{C} \\ \text{with} \quad & z \mapsto z \end{aligned}$$

  It satisfies the diagram given below:

$$\begin{tikzcd}
C_r \arrow[r, dashed, "i"] \arrow[dr, "p|_{C_r}"'] & \mathbb{C} \arrow[d, dashed, "p"] \\
& \mathbb{C} \setminus \{0\}
\end{tikzcd}$$

As a result, the induced homomorphism i^* of i satisfies the diagram

$$\begin{tikzcd}
\pi_1(C_r) \arrow[r, dashed, "i_*"] \arrow[dr, "\left(p|_{C_r}\right)_*"'] & \pi_1(\mathbb{C}) \arrow[d, dashed, "p_*"] \\
& \pi_1(\mathbb{C} \setminus \{0\})
\end{tikzcd}$$

Or equivalently,

$$\begin{tikzcd}
\mathbb{Z} \cong \pi_1(C_r) \arrow[r, dashed, "i_*"] \arrow[dr, "\left(p|_{C_r}\right)_*"'] & \{e\} \arrow[d, dashed, "p_*"] \\
& \mathbb{Z} \cong \pi_1(\mathbb{C} \setminus \{0\})
\end{tikzcd}$$

Therefore, $p_* \circ i_*$ is a zero map since $p_*(e) = 0$, i.e., $(p\,|_{C_r})_*$ is a zero homomorphism.

- Then it's natural to study $p\,|_{C_r} : C_r \to \mathbb{C} \setminus \{0\}$. Construct

$$\begin{cases} q(z) = k \cdot z^n, \quad k := \dfrac{p(r)}{r^n} \text{is a constant} \\ p(z) = a_n z^n + \cdots + a_1 z + a_0 \end{cases}$$

Therefore, $p(r) = q(r)$, and $p|_{C_r}, q|_{C_r} : C_r \to \mathbb{C} \setminus \{0\}$.

- We claim that $p|_{C_r} \simeq q|_{C_r}$ for large r. First construct the mapping

$$\begin{aligned} H: \quad & C_r \times [0,1] \to \mathbb{C} \\ \text{with} \quad & H(z,t) = tp(z) + (1-t)q(z) \\ \text{and} \quad & H(z,0) = q(z), \ H(z,1) = p(z) \end{aligned}$$

  If we want to show H is the homotopy between $p|_{C_r}$ and $q|_{C_r}$, it suffices to show that H is well-defined, i.e., $H : C_r \times [0,1] \to \mathbb{C} \setminus \{0\}$.

  Suppose on the contrary that there exists (z,t) such that

$$(1-t)p(z) + tq(z) = 0, \quad |z| = r, t \in [0,1]$$

  Or equivalently,

$$(1-t)(a_n z^n + \cdots + a_1 z + a_0) + t \cdot kz^n = 0.$$

  Substituting k with $p(r)/r^n$ gives

$$a_n z^n + \cdots + a_1 z + a_0 = t\left(a_{n-1}z^{n-1} + \cdots + a_0 - a_{n-1}\frac{z^n}{r} - \cdots - a_1\frac{z^n}{r^{n-1}} - a_0\frac{z^n}{r^n}\right)$$

  The LHS has leading order n, while the RHS has leading order less or equal to $n-1$. As $r = |z| \to \infty$, $t \to \infty$. Therefore, the equality does not hold in the range $t \in [0,1]$ when r is sufficiently large.

  For this choice of $r = |z|$,

$$H : C_r \times [0,1] \to \mathbb{C} \setminus \{0\}$$

  gives the homotopy $p|_{C_r} \simeq q|_{C_r}$.

- Therefore, we imply $(p|_{C_r})_* = (q|_{C_r})_*$. Now we check the mapping $(q|_{C_r})_* : \mathbb{Z} \to \mathbb{Z}$. In particular, we check the value of $(q|_{C_r})_*(1)$, where 1 is the generator in $\pi_1(C_r)$.

Here we construct the loop

$$\begin{aligned} \ell: \quad & I \to C_r \\ \text{with} \quad & \ell(t) = re^{2\pi it} \end{aligned}$$

and therefore $[\ell] = 1$. It follows that

$$(q|_{C_r})_*(1) = (q|_{C_r})_*([\ell]) = [q|_{C_r}(\ell)] = q(re^{2\pi it}) = k \cdot r^n \cdot e^{2\pi int} \neq 0.$$

Therefore, $(q|_{C_r})_*$ is not a zero homomorphism, i.e., $(q|_{C_r})_* : \mathbb{Z} \cong \pi_1(C_r) \to \pi_1(C\{0\}) \cong \mathbb{Z}$ is the map $1 \mapsto n$, which gives a contradiction.

$\blacksquare$

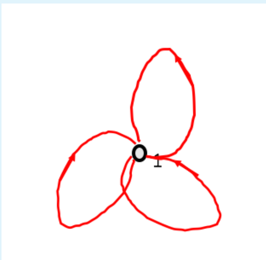
14.3. Monday for MAT4002

14.3.1. Fundamental group of a Graph

Definition 14.3 [Graph] A graph $T = (V, E)$ is defined by the following components:

- V is a finite or countable set, called vertex set;
- E is a finite or countable set, called edge set;
- A function $\delta : E \to V \times V$ with $\delta(e) = (\ell(e), \tau(e))$, where $\ell(e), \tau(e)$ is known as the endpoints of e.

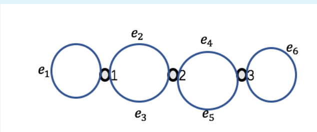
Example 14.2 1. Let $V = \{1\}, E = \{e_1, e_2, e_3\}$, and define $\delta(e_i) = (1,1), i = 1,2,3$. The graph (V, E) is represented below:



2. Let $V = \{e_1, e_2, e_3\}$ and $E = \{e_1, \ldots, e_6\}$, and define

$$\begin{aligned} &\delta(e_1) = (1,1), \quad \delta(e_2) = (1,2), \quad \delta(e_3) = (1,2), \\ &\delta(e_4) = (2,3), \quad \delta(e_5) = (2,3), \quad \delta(e_6) = (3,3). \end{aligned}$$

The graph (V, E) is represented below (We do not care the direction of edges for this graph):

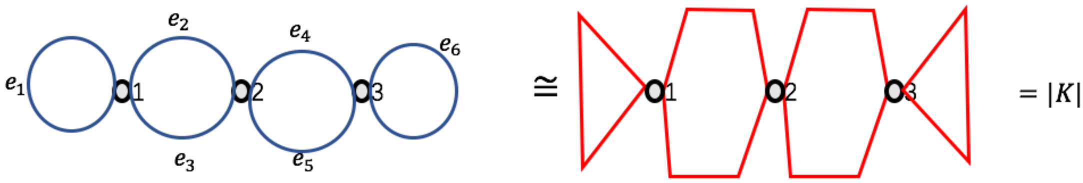


Definition 14.4 [Realizatin of a Graph] For a given graph $\Gamma = (V,E)$, construct a realization by

$$\{|V| \times \{\text{zero simplies}\} \coprod |E| \times \{\text{1-simplies}\}\}/\sim$$

where the equivalence class is induced from the function δ. We still call this realization of the graph as Γ.

(R) In general, graphs are not simplicial complexes. But we can "sub-divide" each edge of Γ into three parts such that there exists simplicial complex K with $|K| \cong \Gamma$. For instance,



where $|K|$ is a simplicial complex.

Definition 14.5

- Subgraph $\Gamma' \subseteq \Gamma$: $\Gamma' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$, and

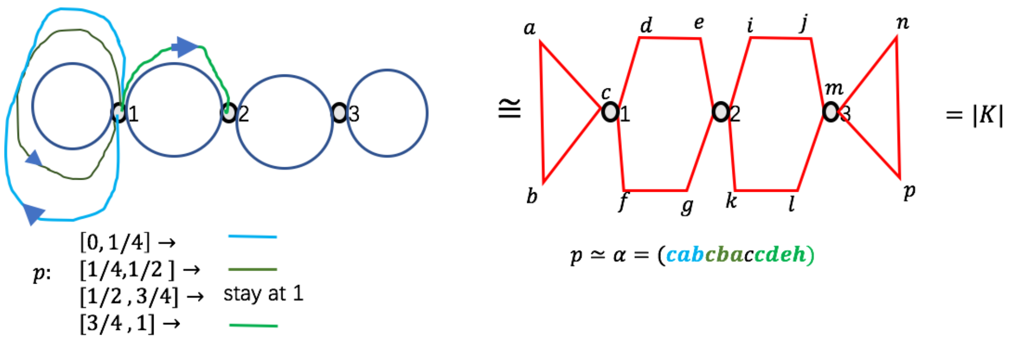
$$\delta\,|_{V'} : E' \to V' \times V'$$

- Edge path: A continous function $p : [0,1] \to \Gamma$ such that there exists $n \in \mathbb{N}$ satisfying

$$p\,|_{[i/n, i+1/n]} : \left[\frac{i}{n}, \frac{i+1}{n}\right] \to T$$

is a path along an edge of Γ, or a constant function on a vertex of Γ, for $0 \le i \le n-1$.

(R) Under the homeomorphism $\Gamma \cong |K|$, each edge path is homotopic to $|g_\alpha|$ for some edge path α in the simplicial complex K. For instance,



- An Edge loop is an edge path p such that $p(0) = p(1) = b \in V$.

- Embedded Edge Loop: An injective edge loop, i.e., $p : [0,1] \to \Gamma$ such that

$$\text{for } x \notin V, \quad p^{-1}(x) = \emptyset \text{ or a single point.}$$

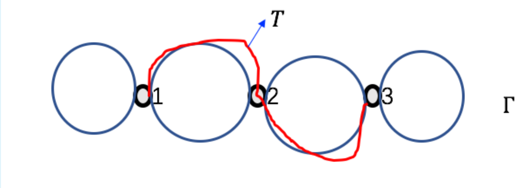
- Tree: a connected graph T that contains no embedded edge loop $p : [0,1] \to T$.

  For instance, as shown in the figure, T_1 contains no edge loop, in particular, the edge loop (a,b,a) is not embedded; T_2 contains embedded edge loop (a,b,c,d,a).

- Maximal Tree of a connected graph Γ:

  - A subgraph T of Γ such that T is a tree.
  - By adding an edge $e \in E(\Gamma) \setminus E(T)$ into T, the new graph is no longer a tree.

  For instance, $T \subseteq \Gamma$ shown in the figure below is a maximal tree.



Theorem 14.5 Let Γ be a connected graph, and T is a subgraph of Γ such that T is a tree. Then T is a maximal tree if and only if $V(T) = V(\Gamma)$.

Moreover, there always exists a maximal tree for all Γ.

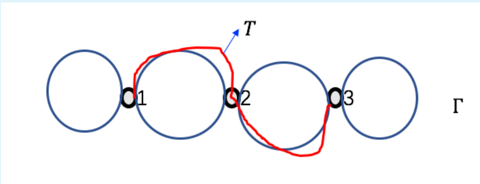
Proof Outline for second part. Construct an ordering of $\{v_1, \ldots, v_i\} \subseteq V(\Gamma)$, such that for each integer $i \geq 2$, there is an edge connecting v_{i+1} with some vertex in $\{v_1, \ldots, v_i\}$.

Then construct $T_1 \subseteq T_2 \subseteq \cdots$, where T_i is a tree containing vertices $\{v_1, \ldots, v_i\}$. As a result, $T = \cup_{i \in \mathbb{N}} T_i$ is a maximal tree. ■

Theorem 14.6 Let Γ be a connected graph. Then $\pi(\Gamma)$ is isomorphic to the free group generated by $\#\{E(\Gamma) \setminus E(T)\}$ elements, for any maximal tree of Γ.

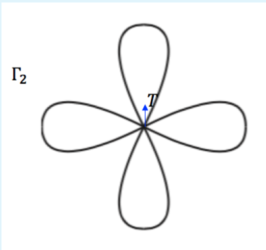
■ **Example 14.3**

1. The graph $T \subseteq \Gamma_1$ shown in the figure below is a maximal tree.



Therefore, $\pi_1(\Gamma_1) \cong \langle a, b, c, d \rangle$ since $\#\{E(\Gamma_1) \setminus E(T)\} = 4$.

2. The graph $T \subseteq \Gamma_2$ shown in the figure below is a maximal tree.



Therefore, $\pi_1(\Gamma_2) \cong \langle a, b, c, d \rangle$ since $\#\{E(\Gamma_2) \setminus E(T)\} = 4$.

3. Note that $\Gamma_1 \simeq \Gamma_2$. The reason for such homotopy equivalence is in the link

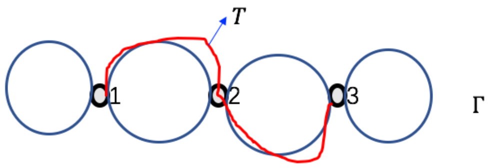
`https://www.math3ma.com/blog/clever-homotopy-equivalences`

■

15.3. Monday for MAT4002

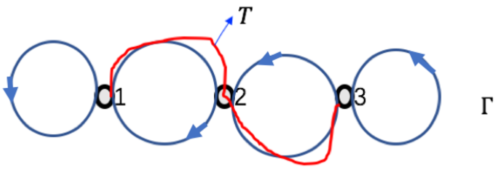
Theorem 15.4 Let Γ be a connected graph. Then $\pi(\Gamma)$ is isomorphic to the free group generated by $\#\{E(\Gamma)\setminus E(T)\}$ elements, for any maximal tree of Γ.

Now we give a proof for this theorem on one special case of Γ:

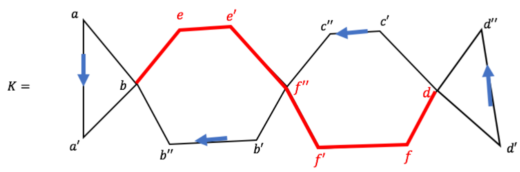


Proof.

- Fix an orientation for each $e\in E(\Gamma)\setminus E(T)$:



- Now let K be a simplicial complex with $|K|\cong\Gamma$:



As a result, $E(K,b)\cong\pi_1(\Gamma)$

- Now we construct the group homomorphism

$$
\begin{aligned}
\phi:\quad & \langle\alpha,\beta,\gamma,\delta\rangle\to E(K,b)\\
\text{with}\quad & \phi(\alpha)=[ba'a''b]\\
& \phi(\beta)=[bee'f''b'b''b]\\
& \phi(\gamma)=[bee'f''f'fdc'c''f''e'eb]\\
& \phi(\delta)=[bee'f''f'fdd''d'dff'f''e'eb]
\end{aligned}
$$

- We can show the group homomorphism ϕ is bijective. In particular, the inverse of ϕ is given by:

$$\Psi: \quad E(K,b) \to \langle \alpha, \beta, \gamma, \delta \rangle$$

where for any $[\ell] := [bv_1 \cdots v_n] \in E(K,b)$, the mapping $\Psi[\ell]$ is constructed by

(a) Remove all other letters appearing in ℓ except $b, a', a'', b', b'', c', c'', d', d''$

(b) Assign

$$\alpha, \ \alpha^{-1}, \ \beta, \ \beta^{-1}, \ \gamma, \ \gamma^{-1}, \ \delta, \ \delta^{-1}$$

for each appearance of

$$a'a'', \ a''a', \ b'b'', \ b''b', \ c'c'', \ c''c', \ d'd'', \ d''d',$$

respectively.

■

15.3.1. The Selfert-Van Kampen Theorem

Theorem 15.5 Let $K = K_1 \cup K_2$ be the union of two **path-connected open** sets, where $K_1 \cap K_2$ is also path-connected. Take $b \in K_1 \cap K_2$, and suppose the group presentations for $\pi_1(K_1, b), \pi_1(K_2, b)$ are

$$\pi_1(K_1, b) \cong \langle X_1 \mid R_1 \rangle, \quad \pi_1(K_2, b) \cong \langle X_2 \mid R_2 \rangle.$$

Let the inclusions be

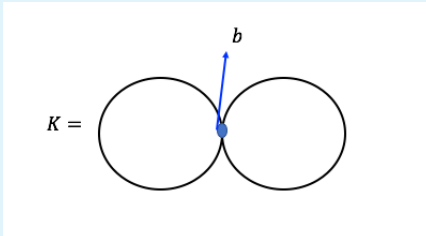
$$i_1 : K_1 \cap K_2 \hookrightarrow K_1, \quad i_2 : K_1 \cap K_2 \hookrightarrow K_2,$$

then a presentation of $\pi_1(K, b)$ is given by:

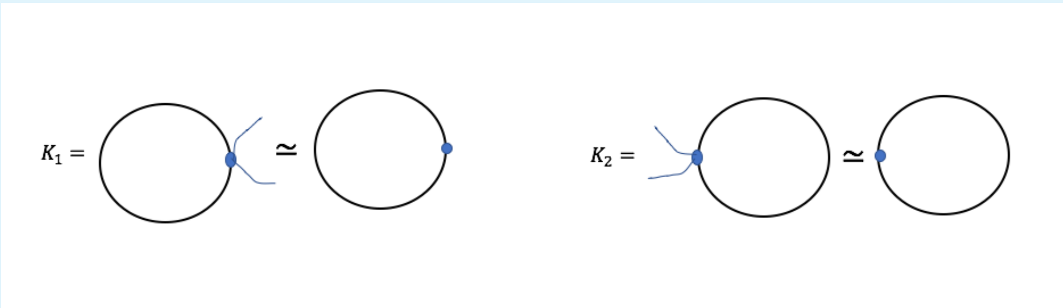
$$\pi_1(K, b) \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{(i_1)_*(g) = (i_2)_*(g) : \forall g \in \pi_1(K_1 \cap K_2, b)\} \rangle.$$

(Here $(i_1)_*: \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_1, b)$ and $(i_2)_*: \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_2, b)$.)

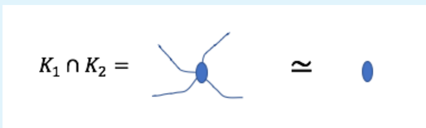
■ **Example 15.4** 1. Let $K = S^1 \wedge S^1$ given by



(a) Then construct b as the intersection between two circles, and construct K_1, K_2 as shown below:



We can see that $K_1 \cap K_2$ is contractible:



(b) As we have shown before, $\pi_1(S^1) \cong \mathbb{Z}$, which follows that

$$\pi_1(K_1, b) \cong \langle \alpha \rangle, \quad \pi_1(K_2, b) \cong \langle \beta \rangle$$

Also, $\pi_1(K_1 \cap K_2, b) \cong \pi_1(\{b\}, b) \cong \{e\}$.

(c) It's easy to compute $(i_1)_*$ and $(i_2)_*$:

$$\begin{array}{ll} (i_1)_*: & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} & e \mapsto e \end{array}, \qquad \begin{array}{ll} (i_2)_*: & \pi_1(K_1 \cap K_2) \to \pi_1(K_2) \\ \text{with} & e \mapsto e \end{array},$$

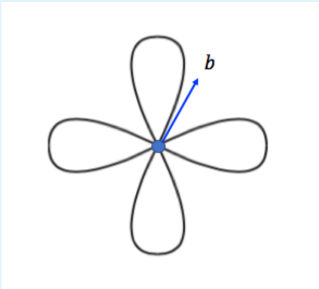
(d) Therefore, by Seifert-Van Kampen Theorem,

$$\pi_1(K,b) \cong \langle \alpha, \beta \mid e = e \rangle \cong \langle \alpha, \beta \rangle$$

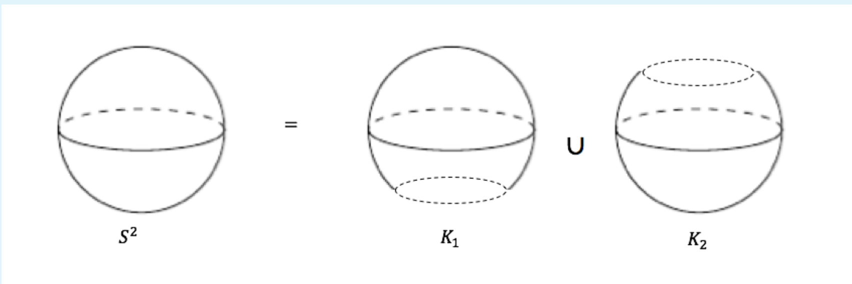
2. By induction,

$$\pi_1(\wedge^n S^1, b) \cong \langle a_1, \ldots, a_n \rangle$$

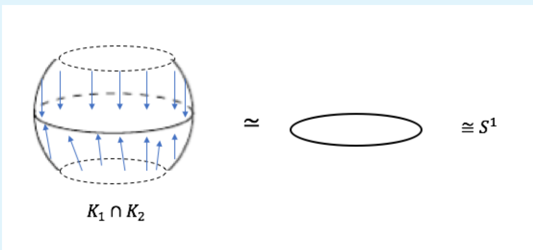
For instance, the figure illustration for $\wedge^4 S^1$ and the basepoint b is given below:



3. (a) Construct $S^2 = K_1 \cup K_2$, which is shown below:



Therefore, we see that $K_1 \cap K_2 \simeq S^1$:



(b) It's clear that K_1 and K_2 are contractible, and therefore

$$\pi_1(K_1) \cong \langle \beta \mid \beta \rangle, \quad \pi_1(K_2) \cong \langle \gamma \mid \gamma \rangle$$

and $\pi_1(K_1 \cap K_2) \cong \pi_1(S^1) \cong \langle \alpha \rangle$.

(c) Then we compute $(i_1)_*$ and $(i_2)_*$. In particular, the mapping $(i_1)_*$ is defined as

$$\begin{aligned} (i_1)_* : \quad & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} \quad & [\alpha] \mapsto [i_1(\alpha)] \end{aligned}$$

where α is any loop based at b. Since K_1 is contractible, we imply α in K_1 is homotopic to c_b, i.e.,

$$(i_1)_*([\alpha]) = [i_1(\alpha)] = e, \forall \alpha \in \pi_1(K_1 \cap K_2).$$

Similarly, $(i_2)_*([\alpha]) = e$.

(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(S^2) \cong \langle \beta, \gamma \mid \beta, \gamma, e = e \rangle \cong \{e\}$$

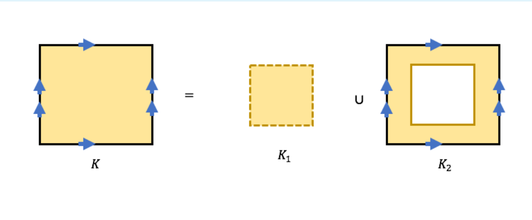
4. Homework: Use the same trick to check that $\pi_1(S^n) = \{e\}$ for all $n \geq 2$. Hint: for S^3, construct

$$K_1 = \{(x_1, \ldots, x_4) \in S^3 \mid x_4 > -1/2\}$$

and

$$K_1 = \{(x_1, \ldots, x_4) \in S^3 \mid x_4 < 1/2\}$$

5. (a) Consider the quotient space $K \cong \mathbb{T}^2$, and we construct $K = K_1 \cup K_2$ as follows:



Therefore, we can see that K_1 is contractible, and K_2 is homotopy equivalent to $S^1 \wedge S^1$:

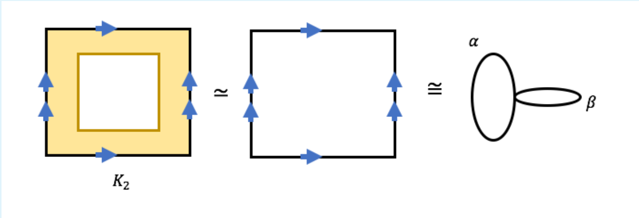
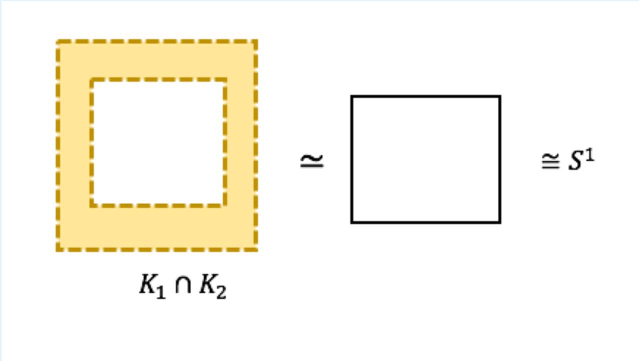


Figure 15.2: Illustration for $K_2 \simeq S^1 \wedge S^1$

and $K_1 \cap K_2$ is homotopic equivalent to the circle:



(b) It follows that

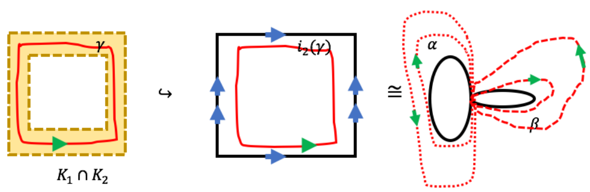
$$\pi_1(K_1) \cong \{e\}, \quad \pi_1(K_2) \cong \langle \alpha, \beta \rangle,$$

and $\pi_1(K_1 \cap K_2) \cong \langle \gamma \rangle$.

(c) Then we compute $(i_1)_*$ and $(i_2)_*$. In particular, $(i_1)_*$ is trivial:

$$\begin{aligned} (i_1)_* : \quad & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} \quad & [\alpha] \mapsto e \end{aligned}$$

Then compute $(i_2)_*$. In particular, for any loop γ, we draw the graph for $i_2(\gamma)$:



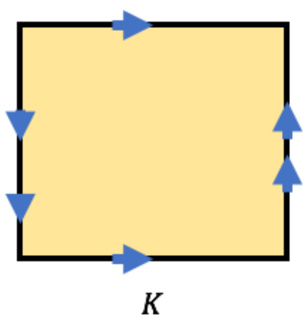
Therefore,

$$(i_2)_*[\gamma] = [i_2(\gamma)] = [\alpha\beta\alpha^{-1}\beta^{-1}]$$

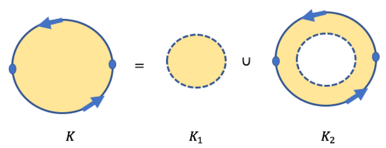
(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha, \beta \mid \beta, \alpha\beta\alpha^{-1}\beta^{-1} = e \rangle \cong \langle \alpha, \beta \mid, \alpha\beta = \beta\alpha \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

6. Exerise: The Klein bottle K shown in graph below satisfies $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$.



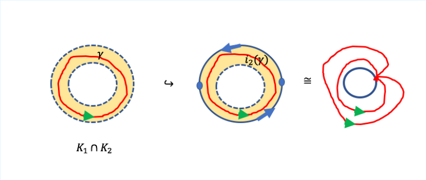
7. Consider the quotient space $K = \mathbb{R}P^2$. We construct $K = K_2 \cup K_2$, which is shown below:



(a) It's clear that K_1 is contractible. In hw3, question 1, we can see that $K_2 \simeq S^1$. Moreover, similar as in (5), $K_1 \cap K_2 \simeq S^1$.

(b) Therefore, $\pi_1(K_1) = \{e\}$ and $\pi_1(K_2) = \langle\alpha\rangle$, $\pi_1(K_1 \cap K_2) = \langle\gamma\rangle$.

(c) It's easy to see that $(i_1)_*([\gamma]) = e$ for any loop γ. For any loop γ, we draw the graph for $i_2(\gamma)$:



Therefore, $(i_2)_*([\gamma]) = [i_2(\gamma)] = [\alpha^2]$.

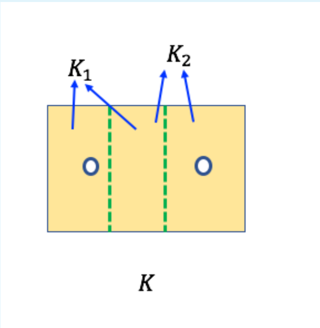
(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle\alpha \mid \alpha^2 = e\rangle \cong \mathbb{Z}/2\mathbb{Z} \cong \{0,1\}_{\text{mod }(2)}$$

8. Let $K = \mathbb{R}^2 \setminus \{2 \text{ points } \alpha, \beta\}$. As have shown in hw3, $K \simeq S^1 \wedge S^1$, which implies

$$\pi_1(K) \cong \pi_1(S^1 \wedge S^1) \cong \langle\alpha, \beta\rangle.$$

We can compute the fundamental group for K directly. Construct $K = K_1 \cup K_2$ as follows:



(a) It's clear that $K_1 \cong \mathbb{R}^2 \setminus \{\text{one point}\} \simeq S^1$ and similarly $K_2 \simeq S^1$. Moreover, $K_1 \cap K_2$ is contractible

(b) Therefore,

$$\pi_1(K_1) \cong \langle\alpha\rangle, \quad \pi_1(K_2) \cong \langle\beta\rangle, \quad \pi_1(K_1 \cap K_2) \cong \{e\}$$

(c) Therefore, $(i_1)_*$ and $(i_2)_*$ is trivial since $\pi_1(K_1 \cap K_2) \cong \{e\}$.

(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha, \beta \mid e = e \rangle \cong \langle \alpha, \beta \rangle$$