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Ordinary Differential Equation

MAT2002 Notebook

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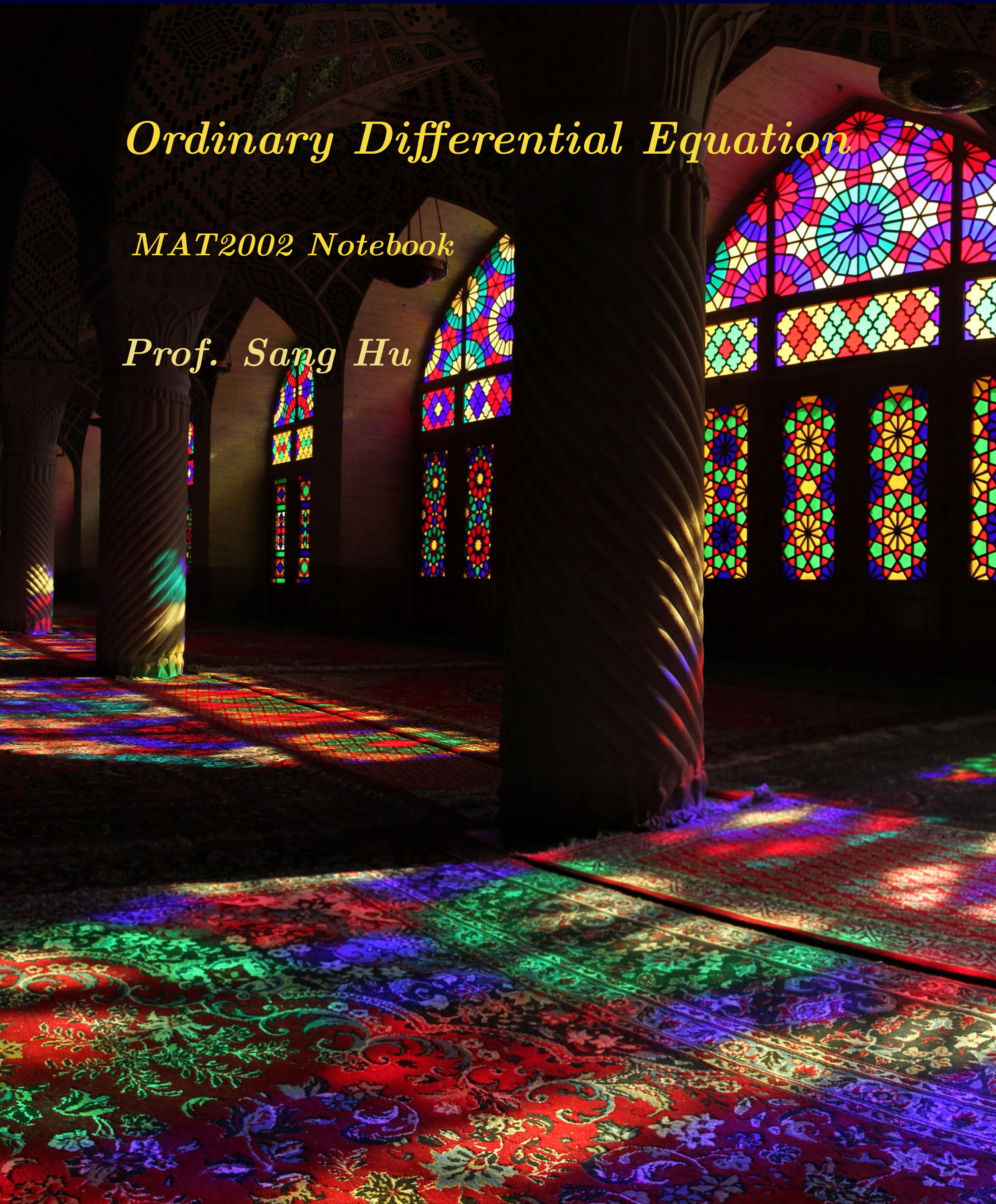

The First Edition

These lecture notes are taken from MAT2002 course in fall of 2017.

Contents

1 — Week1

1.1 Example of mathematical modelling

Exercise 1.1 Put S_0 dollars into a bank. Assume annual interest rate is r, which is *continuously compounded.*
Derive a function that describes the *growth* of the deposit. ■

Solution.

1. Denote $S = S(t)$ as the deposit at time t.
2. Principle: The rate of chance in the deposit at any time is equal to the rate at which interest is calculated:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = rS \qquad \textbf{1st order linear ODE.}$$

3. Initial condition: $S(t=0) = S_0$.
4. Thus we solve the initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = rS \\ S(t=0) = S_0 \end{cases} \implies S(t) = S_0 e^{rt}.$$

We may plot the graph of the solution:

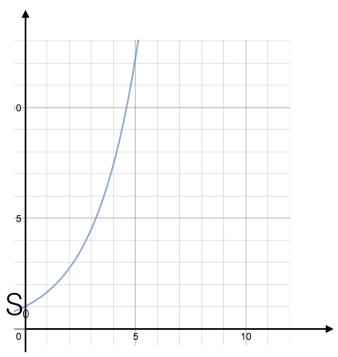

Figure 1.1: Plotting of the solution

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1.1.1 1st order ODE

- Gereral 1st order ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(t,u). \quad \text{where } u = u(t) \text{ is unknown.}$$

- Linear 1st order ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a(t)u(t) + g(t) \tag{1.1}$$

where $a(t), g(t)$ are given **continuous** function.

- How to solve (1.1)?
  - Case 1: Homogeneous ODE, when $g(t) \equiv 0$.

■ Solution 1.1

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a(t)u. \tag{1.2}$$

Assume $u \neq 0$, then we derive

$$\frac{\mathrm{d}u}{u} = a(t)\,\mathrm{d}t.$$

By integrating both sides we find that

$$LHS = \int \frac{\mathrm{d}u}{u} = \ln|u|$$
$$RHS = \int a(t)\,\mathrm{d}t + C \quad \text{for arbitrary constant C.}$$

Thus $\ln|u| = \int a(t)\,\mathrm{d}t + C$.

$$\implies u(t) = C\exp\left(\int a(t)\,\mathrm{d}t\right) \text{ for arbitrary constant C.}$$

■

Example:

For the ODE $\frac{\mathrm{d}u}{\mathrm{d}t} = 2tu$, the solution is given by:

$$u(t) = C\exp\left(\int 2t\,\mathrm{d}t\right) = Ce^{t^2} \text{ for } \forall \text{ constant } C.$$

  - Case 2:Nonhomogeneous ODE, when $g(t) \neq 0$.

■ Solution 1.2 Variation of constant formula:
Conjecture the solution to be

$$u(t) = C(t)\exp\left(\int a(t)\,\mathrm{d}t\right)$$

with $C(t)$ ti be determined.

Then we put this expression into Eq.(1.1) to derive:

$$LHS = C'(t)\exp\left(\int a(t)\,\mathrm{d}t\right) + C(t)a(t)\exp\left(\int a(t)\,\mathrm{d}t\right)$$

$$RHS = g(t) + C(t)a(t)\exp\left(\int a(t)\,\mathrm{d}t\right)$$

which implies

$$C'(t)\exp\left(\int a(t)\,\mathrm{d}t\right) = g(t) \implies C'(t) = g(t)\exp\left(-\int a(t)\,\mathrm{d}t\right)$$

Hence we derive the formula for $C(t)$:

$$C(t) = \int g(t)\exp\left(-\int a(t)\,\mathrm{d}t\right)\mathrm{d}t + C \text{ for } \forall \text{ constant C.}$$

In summary,

$$u(t) = \exp\left(\int a(t)\,\mathrm{d}t\right)\int\left[g(t)\exp\left(-\int a(t)\,\mathrm{d}t\right)\right]\mathrm{d}t \tag{1.3}$$

$$+ C\exp\left(\int a(t)\,\mathrm{d}t\right). \tag{1.4}$$

where (1.3) is the **particular solution** and (1.4) is the solution to homogeneous ODE. (**special solution**) ■

And we have another method to solve this ODE:

■ **Solution 1.3** Integrating factor method:
The key idea is to choose some function $s(t)$ s.t.

$$g(t)s(t) = \frac{\mathrm{d}[us]}{\mathrm{d}t}. \tag{1.5}$$

Then we convert Eq.(1.1) into:

$$g(t) = \frac{\mathrm{d}u}{\mathrm{d}t} - a(t)u$$

And by multiplying factor $s(t)$ both sides we derive:

$$\begin{aligned} g(t)s &= s\frac{\mathrm{d}u}{\mathrm{d}t} - a(t)us \\ &= \frac{\mathrm{d}[us]}{\mathrm{d}t} = s\frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{\mathrm{d}s}{\mathrm{d}t}. \end{aligned}$$

Equivalently,

$$u\frac{\mathrm{d}s}{\mathrm{d}t} = -a(t)us \implies \frac{\mathrm{d}s}{\mathrm{d}t} = -a(t)s \implies s(t) = C\exp\left(-\int a(t)\,\mathrm{d}t\right).$$

If we choose integrating factor to be

$$s(t) = \exp\left(-\int a(t)\,\mathrm{d}t\right),$$

then we integrate Eq.(1.5) both sides:

$$\int \frac{\mathrm{d}[us]}{\mathrm{d}t}\,\mathrm{d}t = \int g(t)s(t)\,\mathrm{d}t. \implies us = \int g(t)s(t)\,\mathrm{d}t + C.$$

Hence the final formula is

$$\begin{aligned} u &= s^{-1}\left[\int g(t)s(t)\,\mathrm{d}t + C\right] \\ &= \exp\left(\int a(t)\,\mathrm{d}t\right)\left[\int g(t)\exp\left(-\int a(t)\,\mathrm{d}t\right) + C\right] \end{aligned}$$

Example:
We tend to solve the **IVP**: (initial value problem)

$$\begin{cases} u' + 2u = e^{-t} \\ u(0) = 1 \end{cases}$$

Solution. $*$ Homogeneous:

$$u' + 2u = 0 \implies u(t) = Ce^{-2t}.$$

$*$ Non-homogeneous:

$$\begin{aligned} u(t) &= e^{-2t}\left(\int e^{-t}\exp(-2t) + C\right) \\ &= e^{-2t}\left(e^{t} + C\right) \end{aligned}$$

Notice that $u(t=0) = 1$, thus we derive:

$$e^0(C + e^0) = 1 \implies C = 0.$$

In conclusion,

$$u(t) = Ce^{-2t} + e^{-t}.$$

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2 — Week2

2.1 Separable equations

Here we give a definition for a special class of first order equations that can be solved by direct integration:

Definition 2.1 — Separable. First order ODEs that have the form as:

$$M(x)+N(y)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

or

$$M(x)\,\mathrm{d}x+N(y)\,\mathrm{d}y=0$$

are said to be **separable**.

Then we show the process to solve *separable* ODEs:

Solution 2.1 For the ODE

$$M(x)+N(y)\frac{\mathrm{d}y}{\mathrm{d}x}=0 \tag{2.1}$$

We define H_1 and H_2 to be any *antiderivatives* of M and N. i.e.

$$\frac{\mathrm{d}H_1}{\mathrm{d}x}=M(x) \qquad \frac{\mathrm{d}H_2}{\mathrm{d}y}=N(y)$$

Thus the Eq.(2.1) becomes

$$H_1'(x)+H_2'(y)\frac{\mathrm{d}y}{\mathrm{d}x}=0. \tag{2.2}$$

We regard y as a function of x and apply the chain rule:

$$H_2'(y)\frac{\mathrm{d}y}{\mathrm{d}x} = \left[\frac{\mathrm{d}}{\mathrm{d}y}H_2(y)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}H_2(y).$$

Consequently, we write Eq.(2.2) as:

$$\frac{\mathrm{d}}{\mathrm{d}x}[H_1(x)+H_2(y)] = 0. \tag{2.3}$$

By integrating the Eq.(2.3) we derive:

$$H_1(x)+H_2(y) = c \text{ for } \forall c \in \mathbb{R}.$$

Or equivalently, the general solution for Eq.(2.1) is given by:

$$\int M(x)\,\mathrm{d}x + \int N(y)\,\mathrm{d}y = c \text{ for } \forall c \in \mathbb{R}.$$

■

Example 1:
Solve the IVT:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1 \end{cases}$$

Solution. We write this ODE as:

$$2(y-1)\,\mathrm{d}y = (3x^2+4x+2)\,\mathrm{d}x$$

Then we do the integration both sides:

$$y^2-2y = x^3+2x^2+2x+c \text{ for } \forall c \in \mathbb{R}.$$

The condition $y(0) = -1$ gives $c = 3$. Hence we obtain the solution:

$$y^2-2y = x^3+2x^2+2x+3.$$

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1. Sometimes the equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \tag{2.4}$$

has a constant solution $y = y_0$. Such a solution is easy to find. For example, the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(y-3)\cos x}{1+2y^2}$$

has the constant solution $y = 3$. Other solutions could be found by separating the variables and do integration.
2. Sometimes it's better to leave teh solution in implicit form. If it's convenient to do so, we usually find the solution explicitly.

Definition 2.2 — Bernoulli Equation. An ODE of the form:

$$y' + q(t)y = \frac{g(t)}{y^{\alpha}}$$

is called the **Bernoulli equation**, where α is arbitrary real number except for $\alpha = 0$ and $\alpha = -1$. ■

■ **Solution 2.2** In order to solve the **Bernoulli equation**:

$$y' + q(t)y = \frac{g(t)}{y^{\alpha}}, \tag{2.5}$$

We use the substitution $v = y^{\alpha+1}$ to derive that:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = (\alpha+1)y^{\alpha}\frac{\mathrm{d}y}{\mathrm{d}t} \implies \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{(\alpha+1)y^{\alpha}}\frac{\mathrm{d}v}{\mathrm{d}t}$$

We put this expression into Eq.(2.5) to obtain:

$$\frac{1}{(\alpha+1)y^{\alpha}}\frac{\mathrm{d}v}{\mathrm{d}t} + q(t)y = \frac{g(t)}{y^{\alpha}}$$

Or equivalently,

$$\frac{\mathrm{d}v}{\mathrm{d}t} + (\alpha+1)q(t)v = g(t).$$

Thus we only need to solve this *linear first order ODE.* ■

Example 2:
Solve the ODE

$$y' = ry - ky^2$$

where $r > 0$ and $k > 0$.

Solution. We use the substitution $v = y^{-1}$ to derive that:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}t} \implies \frac{\mathrm{d}y}{\mathrm{d}t} = -y^2\frac{\mathrm{d}v}{\mathrm{d}t}$$

We put this expression into the original equation:

$$-y^2\frac{\mathrm{d}v}{\mathrm{d}t} = ry - ky^2$$

Or equivalently,

$$\frac{\mathrm{d}v}{\mathrm{d}t} + rv = k.$$

Then we multiply both sides by integrating factor $u(t) = e^{rt}$ to derive:

$$\frac{\mathrm{d}}{\mathrm{d}t}[ve^{rt}] = ke^{rt}$$

Hence

$$ve^{rt} = \int ke^{rt}\,\mathrm{d}t \implies v = \frac{k}{r} + ce^{-rt}$$

Thus the solution to the ODE is:

$$y = \frac{r}{k + ce^{-rt}} \text{ for } \forall c \in \mathbb{R}.$$

■

Definition 2.3 — Homogeneous Equations. The ODE that has the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right) \tag{2.6}$$

is said to be the **homogeneous equation.** ■

Such equations can always be transformed into separable equations by a change of the dependent variable:

■ **Solution 2.3** In order to solve Eq.(2.6), We introduce a new unknown:

$$v(x) = \frac{y(x)}{x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Then we put this expression into Eq.(2.6) to obtain:

$$\frac{\mathrm{d}v}{f(v) - v} = \frac{\mathrm{d}x}{x}$$

Then we do integration both sides to derive:

$$\int \frac{\mathrm{d}v}{f(v) - v} = \int \frac{\mathrm{d}x}{x}$$

Thus the solution is given by:

$$u\left(\frac{y}{x}\right) - \ln|x| = c$$

where $u(s) = \int \frac{\mathrm{d}s}{f(s)-s}$ and c is a constant. ■

Example 3:
Solve the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + xy + y^2}{x^2}$$

Solution. We transform this ODE into homogeneous:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) + 2$$

Thus we introduce $v = \frac{y}{x}$ to derive:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$$

We put this expression into the origin ODE to derive:

$$x\frac{\mathrm{d}v}{\mathrm{d}x}+v=v^2+v+2$$

Or equivalently,

$$\frac{\mathrm{d}v}{v^2+2}=\frac{\mathrm{d}x}{x} \implies \int\frac{\mathrm{d}v}{v^2+2}=\int\frac{\mathrm{d}x}{x}$$

The solution to the ODE is

$$\frac{1}{\sqrt{2}}\int\frac{\mathrm{d}\frac{v}{\sqrt{2}}}{(\frac{v}{\sqrt{2}})^2+1}=\int\frac{\mathrm{d}x}{x} \implies \frac{1}{\sqrt{2}}\arctan(\frac{v}{\sqrt{2}})-\ln|x|=c \text{ for } c\in\mathbb{R}.$$

■

2.2 Difference between linear and nonlinear ODEs

We are familiar with the first order linear ODE. Now let's state a theorem that guarantee the existence and uniqueness of its solution.

Theorem 2.1 — Theorem of uniqueness and existence for first order linear ODE. Given IVP

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases} \tag{2.7}$$

If p, q are c.n.t on an open interval $I = (\alpha, \beta)$ and $t_0 \in I$, then there exists unique $y = y(t)$ that solves Eq.(2.7).

The general first order ODEs have the form:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = f(t,y) \\ y(t = t_0) = y_0 \end{cases} \tag{2.8}$$

And there is a theorem that states the condition when the equation(2.8) has the unique solution:

Theorem 2.2 — Theorem of uniqueness and existence for general first order ODE. For the rectangle

$$\boldsymbol{R} = \{(t,y) : |t - t_0| \le a, \quad |y - y_0| \le b\},$$

We assume that
1. f is *continuous* on $\boldsymbol{R}$.
2. $\frac{\partial f}{\partial y}$ is *continuous* on $\boldsymbol{R}$.

Then there $\exists 0 < h \le a$ s.t. Eq.(2.8) has a unique solution $y = y(t)$ for $t \in (t_0 - h, t_0 + h)$.

We can apply this theorem to solve some ODEs:
Example 5:
Solve the ODE: $\begin{cases} y' = (\sin y)^8 \\ y(0) = 0 \end{cases}$.

Solution. We observe that $y(t) \equiv 0$ is one particular solution.
Since
- $f(t,y) = (\sin y)^8$ is continuous on $\mathbb{R}$;
- $\frac{\partial f(t,y)}{\partial y} = 8(\sin y)^7 \cos y$ is continuous on $\mathbb{R}$,

We derive that $y(t) \equiv 0$ is the unique solution. ■

Sometimes we can determine whether an ODE could be uniquely solved:
Example 6:
Figure out whether the ODE below has an unqiue solution:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{3t^2 + 4t + 2}{2(y-1)} \\ y(0) = -1 \end{cases}$$

Solution.
- $f(t,y) = \frac{3t^2+4t+2}{2(y-1)}$ is c.n.t except for $y = 1$.

- $\frac{\partial f}{\partial y} = -\frac{3t^2+4t+2}{2(y-1)^2}$ is c.n.t. except for $y=1$.
Then an rectangle $\boldsymbol{R}$ containing the point $(0,-1)$ could be drawn s.t.

$$f \text{ and } \frac{\partial f}{\partial y} \text{are c.n.t on } \boldsymbol{R}.$$

Therefore, near point $(0,-1)$, the ODE could be uniquely solved. ■

Example 7:
Figure out whether the ODE below has an unqiue solution:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3t^2+4t+2}{2(y-1)} \\ y(0) = -1 \end{cases}$$

Solution. No rectangle can be drawn around point $(0,1)$. Hence this ODE doesn't have an unique solution.
However, it is *separable*! Thus we obtain:

$$\int 2(y-1)\,\mathrm{d}y = \int (3t^2+4t+2)\,\mathrm{d}t. \implies y^2-2y = t^3+2t^2+2t+C$$

Setting $y(0) = -1$ we obtain:

$$C = -3.$$

Hence we derive:

$$y^2-2y+1 = t^3+2t^2+2t+4 \implies y = 1 \pm \sqrt{t^3+2t^2+2t+4}.$$

Since $y(0) = -1$, we omit the mius condition:

$$y = 1 - \sqrt{t^3+2t^2+2t+4}.$$

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1. *Uniqueness* implies the *trajectories* of any solution to the ODE never **self-intersets** with each other.
2. For *linear* ODE, there is general solutions.
   For *nonlinear* ODE, there may not have general solutions.
3. For *linear* ODE, there exists **explicit solutions**.
   For *nonlinear* ODE, there may not exist **explicit solutions**.

Example 8:
Finite time blow-up: solve the ODE:

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

Solution. The solution is given by:

$$y(t) = \frac{1}{1-t}$$

However, we observe that $y(t) \to \infty$ as $t \to 1$. Hence the theorem (2.2) doesn't guarantee the **global existence of the solution**. ■

What happened? The theorem(2.2) just states the 1st order ODE has a unique solution *on specific intervals* when satisfying its conditions. But this theorem doesn't state *there will exists a global solution!* The theorem only says there will exists a **local** solution on the interval $|t| \leq h$. How to proof the theorem(2.2)?

Proof for existence part. W.L.O.G, suppose $(t_0, y_0) = (0,0)$.
Then we rewrite the ODE(2.8) to be the integration form:

$$y(t) = \int \mathrm{d}y = \int_{t_0=0}^{t} f(s, y(s))\,\mathrm{d}s + y_0$$

We define $\phi_0(t) \equiv 0$ for $|t| \leq h$. (t is to be determined later) And we obtain:

$$\phi_1(t) = \int_{t_0=0}^{t} f(s, \phi_0(s))\,\mathrm{d}s + y_0$$
$$\dots$$
$$\phi_n(t) = \int_{t_0=0}^{t} f(s, \phi_{n-1}(s))\,\mathrm{d}s + y_0$$

If we finally find that there exists a integer k s.t. $\phi_k(t) = \phi_{k+1}(t)$, then $\phi_k(t)$ is one solution to ODE. But in order to achiveve this, we need two conditions to be satisfied:

1. All $\phi_n(t)$ to be well-defined.
2. $\{\phi_n(t)\}$ is uniformly convergent.

Next we show that this two conditions could be satisfied:

- We will show that we can choose h s.t. all $\phi_n(t)$ to be well-defined. i.e.

  $$(s, \phi_n(s)) \in \boldsymbol{R} \text{ for } |s| \leq h \Leftrightarrow |\phi_n(s)| \leq b \text{ for } |s| \leq h.$$

  Since f is continuous, it must be bounded in $\boldsymbol{R}$. Hence there $\exists M$ s.t.

  $$f(t,y) \leq M \text{ for } \forall (t,y) \in \boldsymbol{R}.$$

  Hence the integration $\phi_n(s)$ satisfy the inequality:

  $$|\phi_n(t)| = \left| \int_{t_0}^{t} f(s, \phi_{n-1}(s))\,\mathrm{d}s \right| + y_0 \leq tM$$

  If we choose $|t| \leq \frac{b}{M}$, then we derive:

  $$|\phi_n(t)| \leq \frac{b}{M} \times M = b.$$

  In other words, all $\phi_n(t)$ is well-defined for $h = \min\{\frac{b}{M}, a\}$.
- Then we'd like to show that $\{\phi_n(t)\}$ is **uniformly convergent**. Let's review the definition again:

  Definition 2.4 — Uniformly convergent. For $\forall |t| \leq h$ and $\forall \varepsilon > 0$, there $\exists N(\varepsilon)$ s.t.

  $$|\phi_n(t) - \phi(t)| < \varepsilon$$

  for $\forall n > N$.

  For series, there is a very convenient test for **uniform convergence**, due to *Weierstrass*.

Theorem 2.3 The sequence of function $\phi_n(t)$ defined on a set $\boldsymbol{E}$ satisfies:

$$|\phi_n(t)| \leq M_n \qquad (x \in \boldsymbol{E}, n = 1,2,3,\dots)$$

Then $\sum \phi_n(t)$ converges uniformly on $\boldsymbol{E}$ if $\sum M_n$ converges.

We could use theorem(2.3) to show $\{\phi_n(t)\}$ is uniformly convergent:

- Firstly, since $\frac{\partial f}{\partial y}$ is continuous on $\boldsymbol{R}$, for $\forall(t,y_1),(t,y_2) \in \boldsymbol{R}$, there exists y_3 between y_1 and y_2 s.t.

$$f(t,y_1) - f(t,y_2) = \left[\frac{\partial}{\partial y} f(t,y_3)\right](y_1 - y_2)$$

Obviously, $\frac{\partial f}{\partial y}$ is bounded on $\boldsymbol{R}$, there exists K s.t.

$$\left|\frac{\partial f}{\partial y}(t,y)\right| \leq K \qquad \text{for } \forall t,y \in \boldsymbol{R}$$

Hence we derive:

$$|f(t,y_1) - f(t,y_2)| = \left|\frac{\partial}{\partial y} f(t,y_3)\right| |y_1 - y_2| \leq K|y_1 - y_2|$$

for $\forall(t,y_1)$ and $(t,y_2) \in \boldsymbol{R}$.

- Secondly, we set $y_1 = \phi_n(t)$ and $y_2 = \phi_{n-1}(t)$ to obtain:

$$|f[t,\phi_n(t)] - f[t,\phi_{n-1}(t)]| \leq K|\phi_n(t) - \phi_{n-1}(t)|$$

- Then we use induction to show that

$$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{MK^{n-1}|t|^n}{n!}$$

1. For $n = 1$, we observe that

$$|\phi_1(t)| \leq tM$$

2. Hence for $n = p$, we assume that

$$|\phi_p(t) - \phi_{p-1}(t)| \leq \frac{MK^{p-1}|t|^p}{p!}$$

3. Then for $n = p+1$, we derive:

$$\begin{aligned}
|\phi_{p+1}(t) - \phi_p(t)| &= |\int_{t_0}^{t} \{f[s,\phi_p(s)] - f[s,\phi_{p-1}(s)]\}\, \mathrm{d}s| \\
&\leq \int_{t_0}^{t} |f[s,\phi_p(s)] - f[s,\phi_{p-1}(s)]|\, \mathrm{d}s \\
&\leq K \int_{t_0}^{t} |\phi_p(s) - \phi_{p-1}(s)|\, \mathrm{d}s \\
&\leq K \int_{t_0}^{t} \left(\frac{MK^{p-1}|s|^p}{p!}\right) \mathrm{d}s \\
&\leq \frac{MK^p|t|^{p+1}}{(p+1)!}
\end{aligned}$$

- We could write $\phi_n(t)$ into the form:
$$\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)] = \sum_{i=1}^{n} \Delta\phi_i(t)$$
where $\Delta\phi_i(t) = \phi_i(t) - \phi_{i-1}(t)$ for $i = 1, 2, \dots, n$.
- And we notice that
$$|\Delta\phi_i(t)| \le \frac{MK^{n-1}|t|^n}{n!}$$
And we obtain:
$$\begin{aligned}\sum_{n=1}^{\infty} \frac{MK^{n-1}|t|^n}{n!} &\le \sum_{n=1}^{\infty} \frac{MK^{n-1}h^n}{n!}\\ &\le \frac{M}{k}\sum_{n=1}^{\infty}\frac{(Kh)^n}{n!}\\ &= \frac{M}{k}(e^{Kh}-1)\end{aligned}$$
Hence $\sum \frac{MK^{n-1}|t|^n}{n!}$ converges. By theorem(2.3) we derive that $\sum \Delta\phi_i(t) = \phi_n(t)$ is uniformly convergent.

- Then we show one solution to Eq.(2.8):
We define $\phi(t) = \lim\limits_{n\to\infty} \phi_n(t)$ since $\phi_n(t)$ is *uniformly convergent*. Then we derive:
$$\begin{aligned}\phi(t) = \lim_{n\to\infty}\phi_n(t) &= \lim_{n\to\infty}\int_0^t f[s,\phi_{n-1}(s)]\,\mathrm{d}s\\ &= \int_0^t \lim_{n\to\infty} f[s,\phi_{n-1}(s)]\,\mathrm{d}s\\ &= \int_0^t f[s,\phi(s)]\,\mathrm{d}s.\end{aligned}$$
Hence $\phi(t)$ is the solution to Eq.(2.8).

■

Proof for the uniqueness part. Suppose there are two functions ϕ and ψ satisfying the Eq.(2.8). Then we obtain:
$$\begin{aligned}|\phi(t)-\psi(t)| &= \left|\int_0^t f[s,\phi(s)]\,\mathrm{d}s - \int_0^t f[s,\psi(s)]\,\mathrm{d}s\right| = \left|\int_0^t \{f[s,\phi(s)] - f[s,\psi(s)]\}\,\mathrm{d}s\right|\\ &\le \left|\int_0^t |f[s,\phi(s)] - f[s,\psi(s)]|\,\mathrm{d}s\right|\\ &\le \left|\int_0^t K|\phi(s)-\psi(s)|\,\mathrm{d}s\right|\\ &= \begin{cases} K\displaystyle\int_0^t |\phi(s)-\psi(s)|\,\mathrm{d}s, & 0 \le t \le a,\\ -K\displaystyle\int_0^t |\phi(s)-\psi(s)|\,\mathrm{d}s, & -a \le t \le 0\end{cases}\end{aligned}$$
Then we set $R(t) = \int_0^t |\phi(s)-\psi(s)|\,\mathrm{d}s$, then we observe that
$$\begin{cases} R(0) = 0\\ R(t) \ge 0 \text{ for } t \ge 0.\\ R'(t) \le KR(t) \text{ for } |t| \le a.\end{cases} \tag{2.9}$$

And we can solve the formula(2.9) to derive:

$$e^{-Kt}R(t) \le 0 \implies R(t) \le 0 \text{ for } |t| \le a.$$

Since $R(t) \ge 0$ for $t \ge 0$, we derive $R(t) = 0$ for $0 \le t \le a$.
Similarly, we obtain $R(t) = 0$ for $-a \le t \le 0$.
Hence $R(t) = 0 \implies \phi(t) = \psi(t)$ for $|t| \le a$. Thus the uniqueness of the solution is constructed.
■

3 — Week 3

Any 1st order ODE could be written as:

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

And most first order equations cannot be solved by elementary integration methods, but some ODE such as exact equations could. Let's give the definition for exact equations:

Definition 3.1 — Exact Equations. Let the ODE

$$M(x,y) + N(x,y)y' = 0 \tag{3.1}$$

be given. Suppose that we can identify a function $\psi(x,y)$ s.t.

$$\frac{\partial\psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial\psi}{\partial y}(x,y) = N(x,y)$$

and such that $\psi(x,y) = c$ defines $y = \phi(x)$ *implicitly* as a differentiable function of x. Then the Eq.(3.1) is called to be an **exact** differential equation.

The solution to Eq.(3.1) is $\psi(x,y) = c$, where c is a constant. Let's show how we derive this conclusion:

Solution 3.1 In order to solve the Eq.(3.1), suppose we can pick a function $\psi(x,y)$ s.t.

$$\frac{\partial\psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial\psi}{\partial y}(x,y) = N(x,y)$$

and such that $\psi(x,y) = c$ defines $y = \phi(x)$ *implicitly* as a differentiable function of x. Then we find

$$M(x,y) + N(x,y)y' = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\psi(x, y = \phi(x)).$$

Hence we derive:

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x,y=\phi(x))=0. \implies \psi(x,y=\phi(x))=c \text{ for } \forall c\in\mathbb{R}.$$

■

We have learnt different methods to solve different ODEs so far, and many of them belong to **exact** ODEs:

■ **Example 3.1**

- Linear homogeneous ODEs could be transformed to be exact:
  For the *linear homogeneous* ODE

  $$\frac{\mathrm{d}y}{\mathrm{d}x}+f(x)y=0 \implies \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}+f(x)=0,$$

  we set $M(x,y)=f(x)$ and $N(x,y)=\frac{1}{y}$. Assume there exists $\psi(x,y)$ s.t.

  $$\frac{\partial}{\partial x}\psi(x,y)=f(x) \qquad \frac{\partial}{\partial y}\psi(x,y)=\frac{1}{y}$$

  Hence our $\psi(x,y)$ is given by:

  $$\psi(x,y)=\int f(x)\,\mathrm{d}x+\ln|y|$$

  Thus our solution is given by:

  $$\int f(x)\,\mathrm{d}x+\ln|y|=c \implies y=c\exp\left(-\int f(x)\,\mathrm{d}x\right) \text{ for } c\in\mathbb{R}.$$

- Separable ODEs are exact:
  For the *separable* ODE

  $$M(x)+N(y)\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

  We can pick our $\psi(x,y)$:

  $$\psi(x,y)=\int M(x)\,\mathrm{d}x+\int N(y)\,\mathrm{d}y$$

  Thus our solution is given by:

  $$\int M(x)\,\mathrm{d}x+\int N(y)\,\mathrm{d}y=c \text{ for } c\in\mathbb{R}.$$

■

3.1 Necessary and Sufficient Conditions

The general form of 1st ODE is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0. \tag{3.2}$$

We can use a theorem to test whether Eq.(3.2) is exact:

Theorem 3.1 Let $M, N, \partial_y M, \partial_x N$ be continuous in a rectangle $\boldsymbol{R} = \{\alpha < x < \beta, \sigma < y < \delta\}$. Then Eq.(3.2) is exact iff. $\partial_y M = \partial_x N$ in $\boldsymbol{R}$.

Proof. Sufficiency.
If Eq.(3.2) is *exact*, then there $\exists \psi(x,y)$ such that

$$\partial_x \psi(x,y) = M; \qquad \partial_y \psi(x,y) = N$$

Then we find that

$$\partial_y M = \partial_y \partial_x \psi \qquad \partial_x N = \partial_x \partial_y \psi.$$

Hence $\partial_y \partial_x \psi = \partial_x \partial_y \psi$ for any *second order continuously differentiable* function ψ.
Thus $\partial_y M = \partial_x N$.
Necessity.
We set

$$\psi(x,y) = \int M(x,y)\,\mathrm{d}x + h(y)$$

Hence we obtain:

$$\begin{aligned} \partial_x \psi &= M(x,y) \\ \partial_y \psi &= \partial_y \int M(x,y)\,\mathrm{d}x + h'(y) \end{aligned}$$

We'd like to let

$$\partial_x \psi = M \qquad \partial_y \psi = N$$

Thus

$$N(x,y) = \partial_y \int M(x,y)\,\mathrm{d}x + h'(y) \implies h'(y) = -\partial_y \int M(x,y)\,\mathrm{d}x + N(x,y)$$

Thus

$$h(y) = \int \left[N(x,y) - \partial_y \int M(x,y)\,\mathrm{d}x \right] \mathrm{d}y.$$

We need to check whether the RHS obtain the formula only in terms of y:

$$\begin{aligned} \partial_x h &= \partial_x \int \left[N(x,y) - \partial_y \int M(x,y)\,\mathrm{d}x \right] \mathrm{d}y \\ &= \int \left[\partial_x N(x,y) - \partial_x \int \partial_y M(x,y)\,\mathrm{d}x \right] \mathrm{d}y \\ &= \int \left[\partial_x N(x,y) - \partial_y N(x,y) \right] \mathrm{d}y \\ &= 0. \end{aligned}$$

Then Eq.(3.2) is *exact*. ∎

R If an ODE is *exact*, we know how to find $\phi(x,y)$.

Example 1:

$$2x\,\mathrm{d}x + (y+x)\,\mathrm{d}y = 0$$

Thus $M(x,y) = 2x, N(x,y) = y+x \implies \partial_y M = 0 \neq \partial_x N = 1$.
Thus it is not exact.

Example 2:

$$(y\cos x + 2xe^x)\,\mathrm{d}x + (\sin x + x^2e^y - 1)\,\mathrm{d}y = 0.$$

Solution.

- Firstly, check exact or not!
- Set $\phi(x,y) = \int M(x,y)\,\mathrm{d}x + h(y)$
- General solution is given by:

$$\phi(x,y) \equiv C.$$

■

Example 3:

$$2y\,\mathrm{d}x + x\,\mathrm{d}y = 0.$$

Solution. $\partial_y M = 2 \neq \partial_x N = 1$. Hence it is not exact!
Multiply both sides by $\frac{1}{xy}$:

$$\frac{2}{x}\,\mathrm{d}x + \frac{1}{y}\,\mathrm{d}y = 0.$$

Thus $\partial_y \hat{M} = 0 = \partial_x \hat{N} = 0$. Hence it is exact. ■

In general, suppose Eq.(3.2) is not exact. But is it possible to find an *integrating factor* $\mu(x,y)$ s.t.

$$\mu(x,y)M(x,y)\,\mathrm{d}x + \mu(x,y)N(x,y)\,\mathrm{d}y = 0 \text{ is exact?} \tag{3.3}$$

■ **Solution 3.2** If Eq.(3.3) is exact, then

$$\partial_y[\mu M] = \partial_x[\mu N]. \implies M\partial_y(\mu) + \mu\partial_y(M) = N\partial_x(\mu) + \mu\partial_x(N)$$

Thus we derive:

$$\mu(\partial_y M - \partial_x N) + M\partial_y \mu - N\partial_x \mu = 0.$$

μ is the solution to the 1st order PDE!

- Case 1: $\frac{\partial_y M - \partial_x N}{N}$ depends only on x:
  We set $\mu = \mu(x)$, then $\partial_y \mu = 0$. Hence $\mu(x)$ is the solution to

$$\mu(\partial_y M - \partial_x N) - Nu' = 0.$$

- Case 2: $\frac{\partial_y M - \partial_x N}{M}$ depends only on y:
  We set $\mu = \mu(y)$, then $\partial_x \mu = 0$. Hence $\mu(y)$ is the solution to

$$\mu(\partial_y M - \partial_x N) + Mu' = 0.$$

Example:
Solve the ODE:

$$(3xy + y^2)\,\mathrm{d}x + (x^2 + xy)\,\mathrm{d}y = 0$$

Solution.

- We observe that:

$$\partial_y M = 3x + 2y$$
$$\partial_x N = 2x + y$$

  Since $\partial_y M \neq \partial_x N$, it is not exact.
- Find an integrating factor:

$$\frac{\partial_y M - \partial_x N}{N} = \frac{x+y}{x^2+xy} = \frac{1}{x}$$

  Hence this term depends only on x.
  Set $\mu = \mu(x)$ to find $\mu(x)$
- Multiply both sides by μ:

$$\mu(3xy + y^2)\,\mathrm{d}x + \mu(x^2 + xy)\,\mathrm{d}y = 0$$

  is exact!

Thus the final solution is given by:

$$x^3 y + \frac{1}{2}x^2 y^2 = C$$

∎

4 — Week 4

4.1 Autonomous 1st order ODE

We study a kind of 1st order ODE that it doesn't show us the independent variable **explicitly**:

> **Definition 4.1 — Autonomous 1st order ODE.**
> The autonomous 1st order ODE has the form:
>
> $$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \tag{4.1}$$
>
> Note that this ODE doesn't show the independent variable $\boldsymbol{t}$. ■

Recall that we have considered the special case of Eq.(4.1) in which $f(y) = ay + b$.

We want to use geometrical methods to find the *qualitative* information directly from the ODE without solving the equation.

We want to find some constant functions $y = c$ s.t. there is **no change** or **variation** in the value of y as t increases. How to find these functions? We just need to let $f(y) = 0$ to derive the roots:

> **Definition 4.2 — Critical point.**
> A **critical point** of a function $y = g(x)$ is a **value** x_0 such that
>
> $$g'(x_0) = 0.$$
>
> Moreover, we define the *critical points* of the Eq.(4.1) to be the zeros of $f(y)$. ■

> **Definition 4.3 — Equilibrium solution.** Suppose y_0 is the critical point of Eq.(4.1), then we derive $f(y_0) = 0$. And we observe that $y = y_0$ is also the solution to Eq.(4.1). Hence $y = y_0$ is called the **Equilibrium solution** to Eq.(4.1). ■

Now we show that we can **visualize other solutions** of Eq.(4.1) and **sketch** their graphs quickly by studying $f(y)$:

■ **Example 4.1** We tend to sketch the general graph of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r(1 - \frac{y}{K})y \text{ for constans } r \text{ and } K. \tag{4.2}$$

We set $f(y) = r(1 - \frac{y}{K})y$ and draw the graph of $f(y)$:

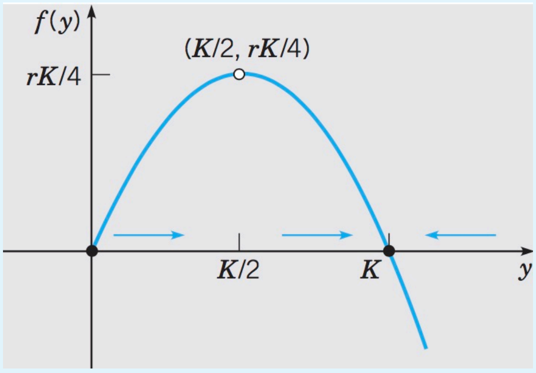


Figure 4.1: $f(y)$ versus y for Eq.(4.2)

In this graph we find the critical points are:

$$y_1 = 0 \qquad y_2 = K$$

And the equilibrium solutions are:

$$y = 0, \qquad y = K.$$

- Observe that $\frac{\mathrm{d}y}{\mathrm{d}t} > 0$ for $0 < y < K$ and $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$ for $y > K$. So we can draw the **phase line** (the y-axis) to show the variation of y. (Shown in Figure(4.2a)).
- We can plot the curves for Eq.(4.2) from the information given below: (The graph is shown in Figure(4.2b)).

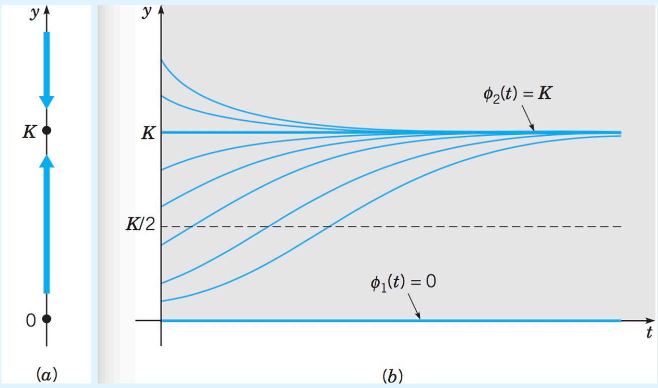


Figure 4.2: The graph for ODE(4.2)

– We can infer from Figure(4.1) that when y is around 0 or K, the value for $f(y)$ is near **zero**. Hence the corresponding curves for Eq.(4.2) are very *flat*. Similarly, when y leaves the neighborhood of 0 and K, the corresponding curves for Eq.(4.2) are very *steep*.
– To draw the Figure(4.2b), we start with the *equilibrium solutions* $y = 0$ and $y = K$; then we draw other curves that are *increasing* for $0 < y < K$ and *decreasing* for $y > K$. And the curves are *flat* when y approaching 0 or K.
– Note that other solutions **cannot** interset with the solution $y = K$. Why? By the *existence and uniqueness theorem*, there should be a **unique** solution that could pass through a given point in the $ty-$plane.
– We can also determine the concavity of those curves by computing $\frac{d^2y}{dt^2}$:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = f'(y)f(y)$$

Interval of y	$(0, \frac{K}{2})$	$(\frac{K}{2}, K)$	$(K, +\infty)$
sign for $f'(y)$	$+$	$-$	$-$
sign for $f(y)$	$+$	$+$	$-$
Concavity	$+$	$-$	$+$

– Finally, we observe that all curves seem to approach K as t increases.

The information above is obtained without solving the ODE. However, sometimes we need a detailed description by solving the ODE:

Example 4.2 Given the condition $y(t = 0) = y_0$ and provided that $y \neq 0$ and $y \neq K$, we transform Eq.(4.2) into form:

$$\frac{dy}{(1 - \frac{y}{K})y} = r\,dt \implies \left(\frac{1}{y} + \frac{\frac{1}{K}}{1 - \frac{y}{K}}\right) dy = r\,dt.$$

By integrating both sides we obtain:

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + c \text{ for constant } c.$$

- When $0 < y_0 < K$, we have known that $0 < y < K$ all the time. Hence we break the absolute value bars to find:

$$\frac{y}{1 - \frac{y}{K}} = Ce^{rt} \text{ for constant } C.$$

By setting $y(t = 0) = y_0$ we derive:

$$C = \frac{y_0}{1 - \frac{y_0}{K}} \implies y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

- Similarly, when $y_0 > K$ we derive the same result.

And we notice that when $y_0 = 0$, by Eq.(4.2), $y = 0$ is the solution; when $y_0 > 0$, if we let $t \to \infty$, we obtain:

$$\lim_{t\to\infty} y(t) = \frac{y_0 K}{y_0} = K$$

Thus for $\forall y_0 > 0$, the solution approaches the *equilibrium solution* $y = K$ **asymptotically** as $t \to \infty$. ■

In this example we notice that for $\forall y_0 > 0$, the solution approaches the equilibrium solution $y = K$ *asymptotically* as $t \to \infty$. We say that $y = K$ is an *asympotically stable solution* of Eq.(4.2):

Definition 4.4 — asympotically stable solution. For the solution to the *autonomous 1st order ODE*(4.1), if the nearby curves all **converges** to the *equilibrium solution* as t increases, then the equilibrium solution is said to be the **asympotically stable solution.** ■

On the other hand, even the solutions for Eq.(4.2) starts very close to zero, it approaches to K as t increases instead of approaching to 0. We say that $y = 0$ is an *unstable equilibrium solution* of Eq.(4.2).

Definition 4.5 — Unstable equilibrium solution. For the solution to the *autonomous 1st order ODE*(4.1), if the nearby curves all **diverge away** from the *equilibrium solution* as t increases, then the equilibrium solution is said to be the **unstable equilibrium solution** ■

Exercise 4.1 Consider the equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

where $f(y) = y(y-1)(y-2)$. Determine the *critical (equilibrium) points* and classify it as *asymptotically stable or unstable*. Sketch several graphs of solutions in the ty-plane. ■

■ **Solution 4.1** Firstly, we draw the graph of f(y):

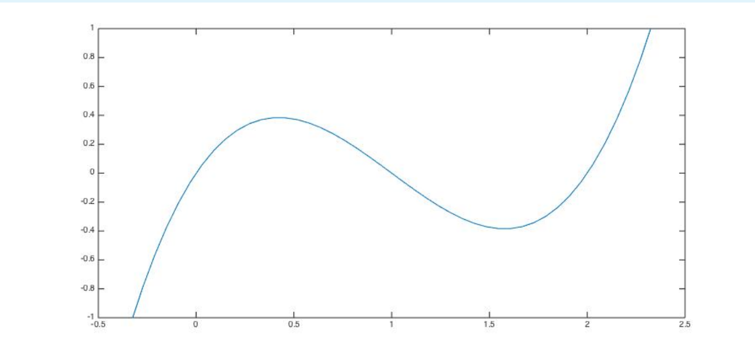


Figure 4.3: $f(y)$ versus y

In this graph we find the critical points are:

$$y_1 = 0 \qquad y_2 = 1 \qquad y_3 = 2.$$

Hence the equilibrium solutions are:

$$y = 0 \qquad y = 1 \qquad y = 2.$$

- Also, we find that:

$$\frac{dy}{dt} < 0 \text{ when } y < 0$$
$$\frac{dy}{dt} > 0 \text{ when } 0 < y < 1$$
$$\frac{dy}{dt} < 0 \text{ when } 1 < y < 2$$
$$\frac{dy}{dt} > 0 \text{ when } y > 2$$

Thus we draw the *phase line* (the y-axis) to show the variation of y.

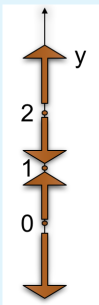


Figure 4.4: The phase line

- We can also determine the concavity of those curves by computing $\frac{d^2y}{dt^2}$:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = f'(y)f(y) = y(y-1)(y-2)(3y^2-6y+2)$$

Interval of y	$(-\infty, 0)$	$(0, 1-\frac{\sqrt{3}}{3})$	$(1-\frac{\sqrt{3}}{3}, 1)$
sign for $f'(y)$	$+$	$+$	$-$
sign for $f(y)$	$-$	$+$	$+$
Concavity	$-$	$+$	$-$

Interval of y	$(1, 1+\frac{\sqrt{3}}{3})$	$(1+\frac{\sqrt{3}}{3}, 2)$	$(2, +\infty)$
sign for $f'(y)$	$-$	$+$	$+$
sign for $f(y)$	$-$	$-$	$+$
Concavity	$+$	$-$	$+$

Finally, we plot the curves from the information given above:

■

Definition 4.6 — 2nd order linear ODE. The general form of 2nd order **linear** ODE is given by:

$$y'' + p(t)y' + q(t)y = g(t) \tag{4.3}$$

If the RHS in Eq.(4.3) is **zero**, it is said to be **homogeneous**, otherwise the Eq.(4.3) is said to be **nonhomogeneous**.
We begin to talk about homogeneous equations first:

4.2 *Homogeneous with constant coefficient*

In this lecture we only talk about the equations in which the functions $p(t)$,$q(t)$ are constants. In this case, the Eq.(4.3) become:

$$ay'' + by' + cy = 0. \tag{4.4}$$

which is said to be the **homogeneous 2nd order linear ODE with constant coefficient**.
Then we try to solve this ODE:

Solution 4.2

- We guess the solution to the Eq.(4.4) to have the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$, $y'' = r^2e^{rt}$. By substituting these expressions for y,y' and y'' we obtain:

  $$(ar^2 + br + c)e^{rt} = 0.$$

  Since $e^{rt} \neq 0$, we derive that

  $$ar^2 + br + c = 0.$$

  Definition 4.7 For the ODE $ay'' + by' + cy = 0$, its **characteristic equation** is given by:

  $$ar^2 + br + c = 0. \tag{4.5}$$

  We can see that if r is a root to Eq.(4.5), then $y = e^{rt}$ is a solution to the ODE(4.4).

- We set $\Delta = b^2 - 4ac$, and discuss three cases to solve this ODE:
  - $\Delta > 0$: There exists two real roots $r_1 \neq r_2$:

    $$r_1 = \frac{-b + \sqrt{\Delta}}{2a} \qquad r_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

    Then the fundamental set of solutions to Eq.(4.4) is

    $$\{y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}\}$$

The general solution is given by:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \text{ for constants } c_1, c_2.$$

- $\Delta < 0$: There exists two complex roots $r_1 \neq r_2$:

$$r_1 = \frac{-b + i\sqrt{|\Delta|}}{2a} = \lambda + i\mu \qquad r_2 = \frac{-b - i\sqrt{|\Delta|}}{2a} = \lambda - i\mu$$

Then the fundamental set of solutions to Eq.(4.4) is:

$$\{y_1 = e^{r_1 t} = e^{\lambda t}(\cos\mu t + i\sin\mu t), \quad y_2 = e^{r_2 t} = e^{\lambda t}(\cos\mu t - i\sin\mu t)\} \tag{4.6}$$

We want to express the solution of such a problem in terms of **real-valued** functions. And we can use a theorem to solve this task: (verify this theorem by yourself)

Theorem 4.1 Consider the Eq.(4.3),

$$y'' + p(t)y' + q(t)y = 0$$

where p and q are *continuous real-valued* functions. If $y = u(t) + iv(t)$ is a *complex-valued* solution of Eq.(4.3), then its **real** part $u(t)$ and its **imaginary** part $v(t)$ are also solutions of this equation.

Thus from the formula(4.6), we obtain a new **real** fundamental set of solutions to Eq.(4.4):

$$\{y_1 = e^{\lambda t}\cos(\mu t), \quad y_2 = e^{\lambda t}\sin(\mu t)\}$$

Hence the general solution is given by:

$$y(t) = e^{\lambda t}[c_1\cos(\mu t) + c_2\sin(\mu t)] \text{ for constants } c_1, c_2.$$

- $\Delta = 0$: There exists two repeated roots r_1, r_2 :

$$r_1 = r_2 = r = -\frac{b}{2a}$$

Obviously, $y_1 = e^{rt}$ is a solution. How to find a second solution? We set

$$y_2 = v(t)y_1(t) = v(t)e^{rt}.$$

Then it follows that

$$y_2' = [rv(t) + v'(t)]e^{rt}$$

and

$$\begin{aligned} y_2'' &= r[rv(t)e^{rt} + v'(t)e^{rt}] + rv'(t)e^{rt} + v''(t)e^{rt} \\ &= [r^2 v(t) + 2rv'(t) + v''(t)]e^{rt} \end{aligned}$$

By substituting these expressions for y, y' and y'' in Eq.(4.4) we obtain:

$$v''(t) = 0 \implies v(t) = c_1 + c_2 t.$$

Hence the second solution could be

$$y_2 = c_1 e^{rt} + c_2 t e^{rt}.$$

In conclusion, the fundamental set of solutions to Eq.(4.4) is given by

$$\{y_1 = e^{rt}, \quad y_2 = te^{rt}\}$$

Hence the general solution is

$$y(t) = e^{rt}(c_1 + c_2 t) \text{ for constants } c_1, c_2.$$

■

5 — Week5

5.1 Theorem of existence and uniqueness

The fundamental theoretical result for IVP for second order **linear** equations is stated below, which is analogous for first order linear equations:

Theorem 5.1 — Existence and Uniqueness Theorem for 2nd order linear equations. 5.1
Consider the IVP

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0' \tag{5.1}$$

where p, q and g are **continuous** on an **open interval** I that contains the point t_0. Then there exists a **unique** solution $y = \phi(t)$ of this problem in the interval I.

R This theorem derives three statements:

- The IVP **exists** a solution.
- The solution to this IVP is **unique**
- The solution ϕ is defined on the interval I and is at least twice differentiable.

Proof. This proof need to be filled ■

■ **Example 5.1** Find the unique solution of the IVP

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = 0, \qquad y'(t_0) = 0$$

where p and q are *continuous* in an *open* interval I containing t_0 ■

Solution: We find that $y = \phi(t) = 0$ is one solution. By the *uniqueness* part of Theorem(5.1), it is the only solution to the given IVP.

5.2 Principle of superposition

Next, we find that if y_1 and y_2 are two solutions to the ODE:

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{5.2}$$

then its **linear combinations** are also the solution to Eq.(5.2). We state this result as a theorem:

Theorem 5.2 — Principle of Superposition. If y_1 and y_2 are two solutions of the Eq.(5.2),

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for constants c_1, c_2.

Proof. We observe that

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= [c_1y_1 + c_2y_2]'' + p(t)[c_1y_1 + c_2y_2]' + q(t)[c_1y_1 + c_2y_2] \\ &= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2 \\ &= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] \\ &= c_1L[y_1] + c_2L[y_2]. \end{aligned}$$

Since y_1 and y_2 are solutions to Eq.(5.2), we derive

$$L[y_1] = L[y_2] = 0 \implies L[c_1y_1 + c_2y_2] = 0.$$

Hence $c_1y_1 + c_2y_2$ is also a solution for constants c_1, c_2. ■

5.3 Wronskian

The next question is that *whether there may be other solutions of a different form from* $c_1y_1 + c_2y_2$? We answer this question in the following steps:

1. Firstly, we claim that $c_1y_1 + c_2y_2$ could be possible solution to the IVP problem if and only if its *Wronskian* is not zero at initial point:
Suppose y_1 and y_2 are gien two solutions to Eq.(5.2), then $c_1y_1 + c_2y_2$ would be possible solution to the IVP problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = 0, \qquad y'(t_0) = 0 \tag{5.3}$$

if and only if we can choose (c_1, c_2) satisfying

$$\begin{cases} (c_1y_1 + c_2y_2)(t_0) = c_1y_1(t_0) + c_2y_2(t_0) = y_0, \\ (c_1y_1 + c_2y_2)'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'. \end{cases} \tag{5.4}$$

The determinant of coefficients of the system(5.4) is

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \tag{5.5}$$

- If $W \neq 0$, then the system(5.4) has a unique solution for (c_1, c_2), according to the *Crema's Rule*, the solution is given by:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}.$$

  In this case, $c_1y_1 + c_2y_2$ could be the possible solution.
- If $W = 0$, there may have *infinitely many* (c_1, c_2) or no (c_1, c_2) to satisfy the system(5.4). Hence in this case, there may not exist the solution in form of $c_1y_1 + c_2y_2$.

During the discussion above, we introduce a special determinant: *Wronskian*.

Definition 5.1 — Wronskian. For the IVP

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \qquad y'(t_0) = y'_0$$

If y_1 and y_2 are two solutions of this problem, then the **Wronskian determinant /Wronskian** of solutions y_1 and y_2 is defined as:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}$$

The *Wronskian* is always useful. It can help us to make sure whether *there exists one linear combination of y_1 and y_2 that could be the possible solution to the IVP problem.* We can conclude the discussions above into a theorem:

Theorem 5.3 Suppose y_1 and y_2 are two solutions of the ODE:

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Given the IVP

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \qquad y'(t_0) = y'_0,$$

it is always possible for us to pick c_1 and c_2 s.t.

$$y = c_1y_1(t) + c_2y_2(t)$$

is the solution to this problem if and only if the *Wronskian* of the solutions y_1 and y_2 at t_0

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}$$

is not zero.

2. Moreover, uner the assumption that p, q are continuous functions, and using the theorem given above, we could derive that the solution to the Eq.(5.2) **only** has the form $c_1y_1 + c_2y_2$ if and only if there exists a point t_0 s.t.

$$W(y_1, y_2)(t_0) \neq 0.$$

Theorem 5.4 Suppose that y_1 and y_2 are two solutions of the Eq.(5.2),

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Then the linear combination of y_1 and y_2,

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constants c_1 and c_2 are the **only** solution to Eq.(5.2) if and only if there exists a point t_0 s.t.

$$W(y_1, y_2)(t_0) \neq 0.$$

5.4 *Fundamental set of solutions*

Can we use a term to describe that the linear combination of y_1 and y_2 are the only solution to Eq.(5.2)? We introduce the definition for **fundamental set of solutions**:

Definition 5.2 — Fundamental set of solutions. Suppose $\{y_1, y_2\}$ is a set of solutions to the ODE

$$y'' + p(t)y' + q(t)y = 0, \tag{5.6}$$

then $\{y_1, y_2\}$ is said to be a **fundamental set of solutions** if it satisfies:

1. y_1 and y_2 solves for Eq.(5.6)
2. If y_0 solves for Eq.(5.6), then it is a linear combination of y_1 and y_2.

Combining the Theorem(5.3) and (5.4) with the definition above, we derive a useful fact:

Theorem 5.5 $\{y_1, y_2\}$ is a fundamental set of solutions to Eq.(5.6) iff.

$$\exists t_0 \text{ such that } W(y_1, y_2)(t_0) \neq 0$$

Linear independence and Wronskian

And we also want to explore more properties for the fundamental set of solutions. Firstly, let's introduce the definiton for linear dependence with respect to functions:

5.5 Linear independence

Definition 5.3 — Linear independence. A set of functions $\{f_1, f_2, \dots, f_n\}$ is called **linear independent** iff.

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0 \Longleftrightarrow k_1 = k_2 = \cdots = k_n = 0.$$

In fact, we can use linear independence to define fundamental set again:

Definition 5.4 — Equivalent definiton for fundamental set of solutions.
A set of solutions $\{y_1, y_2\}$ to Eq.(5.6) is a **fundamental set of solutions** iff.

1. y_1 and y_2 solves for Eq.(5.6)
2. $\{y_1, y_2\}$ are linearly independent.

The proof for this equivalent definition is omitted due to the length of this book.

5.6 Abel's theorem

And we want to know the relationship between Wronskian and fundamental set of solutions, which is obtained from **Abel's theorem**:

Theorem 5.6 — Abel's theorem. If y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{5.7}$$

where p, q are c.n.t. on an open interval I, then the Wronskian is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)\,\mathrm{d}t\right]$$

where c is a certain constant that depends on y_1 and y_2, but not on I. Further more, $W(y_1, y_2)(t)$ is either **zero** or **never zero** for $\forall t \in I$.

Proof. Since y_1 and y_2 are the solutions to Eq.(5.7), we have

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \tag{5.8}$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \tag{5.9}$$

And we let Eq.(5.8)$\times(-y_2)$+Eq.(5.9)$\times(y_1)$ to derive

$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0 \tag{5.10}$$

Then we let $W(t) = W(y_1, y_2)(t) = y_1 y_2' - y_1' y_2$ and we observe that

$$W' = y_1 y_2'' - y_1'' y_2$$

Then we could rewrite Eq.(5.10) into:

$$W' + p(t)W = 0 \implies W(t) = c\exp\left[-\int p(t)\,\mathrm{d}t\right] \text{ where } c \text{ is a constant.}$$

Since the exponential is never zero, the wronskian is either zero or never zero for $\forall t \in I$. ■

5.7 *Completeness of fundamental set of solutions*

In fact, every solution to the 2nd order linear ODE must be uniquely expressed as a lienar combination fo fundamental set of solutions. This is called the completeness of fundamental set of solutions.

Theorem 5.7 — Completeness for fundamental set of solutions. If $\{y_1, y_2\}$ is a fundamental set of solutions to the equation

$$y'' + p(t)y' + q(t)y = 0 \tag{5.11}$$

then every solution to Eq.(5.11) could be uniquely expressed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

where c_1, c_2 are constants.

Before the proof, we have a review of the property about the fundamental set of solutions:

- Property 1: [Check for Theorem(5.6)]
  $W(y_1, y_2)$ is either zero or never zero for $\forall t \in I$.
- Property 2: [Check for Theorem(5.5)]
  $\{y_1, y_2\}$ is fundamental set of solutions iff. $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

We can use these two properties to proof this theorem:

Proof. Suppose $y(t)$ is a solution to Eq.(5.11), since it is well-defined, we set

$$y(t_0) = y_0, \quad y'(t_0) = y'_0$$

Then we intend to find c_1, c_2 s.t.

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y'_1(t_0) + c_2 y'_(t_0) = y'_0 \end{cases} \tag{5.12}$$

By property(2), $W(y_1, y_2) \neq 0$. Hence Eq.(5.12) admits unique pair (c_1, c_2).
We set $\phi(t) = c_1 y_1 + c_2 y_2$. You can verify that

- $\phi(t)$ solves for Eq.(5.11)
- $\phi(t)$ satisfies the same initial condition as $y(t)$.

By the *existence and uniqueness theorem*,

$$\phi(t) = y(t) = c_1 y_1 + c_2 y_2$$

■

R You maybe confued about too many theorems and massy relationships this section. But at least you should hold on these points below:

$\{y_1, y_2\}$ is fundamental set of solutions iff. $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

Every solution to Eq.(5.11) could be uniquely expressed as linear combination of fundamental set of solutions

Abel's theorem

6 — Week6

6.1 Method of reduction of order

In order to solve the 2nd orde linear ODE, we usually use the **reduction of order method:**

- If we know $y_1(t)$ is one solution, we guess the another solution is given by:

$$y_2(t) = v(t)y_1(t)$$

And then we put $y_2(t)$ into our origin ODE to derive the formula of $v(t)$.

Firstly, let's restrict our ODE into **constant coefficient equation** to show how to use this method:

6.2 Constant coefficient equation

In the lecture before, maybe you are confused about why the second solution to the *two repeated roots* case is te^{rt}, which is not obvious. So let's have a brief review on how to derive this solution again:

■ **Solution 6.1** We intend to solve the **homogeneous 2nd order ODE with constant coefficients**:

$$ay'' + by' + cy = 0 \tag{6.1}$$

where its *characteristic equation* has two **repeated** roots.

In this case, $b^2 - 4ac = 0 \implies r = -\frac{b}{2a}$. Hence one solution to Eq.(6.1) is

$$y_1(t) = e^{rt}$$

To find a second solution, we guess

$$y = v(t)y_1(t) = v(t)e^{rt}$$

It follows that

$$y' = [v'(t) + rv(t)]e^{rt}$$

and

$$y'' = [v''(t) + 2rv'(t) + r^2 v(t)]e^{rt}$$

By substituting the above formulas in Eq.(6.1), we obtain:

$$\left\{a[v''(t) + 2rv'(t) + r^2 v(t)] + b[v'(t) + rv(t)] + cv(t)\right\} e^{rt} = 0. \tag{6.2}$$

Or equivalently,

$$[av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t)]e^{rt} = 0 \tag{6.3}$$

Since $e^{rt} \neq 0$, we obtain:

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0.$$

Since $r = -\frac{b}{2a}$ and $b^2 - 4ac = 0$, it's trival that $2ar + b = 0$ and $ar^2 + br + c = 0$. Hence

$$av''(t) = 0 \implies v''(t) = 0. \implies v(t) = c_1 + c_2 t$$

Hence the second solution is given by:

$$y(t) = (c_1 + c_2 t)e^{rt}$$

Thus partly some solutions to Eq.(6.1) are a linear combination of

$$y_1 = e^{rt} \qquad y_2 = te^{rt}$$

And the wronskian is given by:

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & (1 + rt)e^{rt} \end{vmatrix} = e^{rt} \neq 0 \tag{6.4}$$

Hence $\{y_1 = e^{rt}, y_2 = te^{rt}\}$ is a fundamental set of solutions to Eq.(6.1). And the general solution to Eq.(6.1) is

$$y(t) = c_1 e^{rt} + c_2 te^{rt}.$$

■

And also, the reduction of order method could be extended more:

6.3 *General homogeneous ODE*

Suppose $y_1(t)$ is a solution, which is not everywhere zero, of the **homogeneous** ODE:

$$y'' + p(t)y' + q(t)y = 0. \tag{6.5}$$

■ **Solution 6.2** In order to find a second solution, we let

$$y = v(t)y_1(t)$$

It follows that

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

and

$$y'' = [v''(t)y_1(t) + v'(t)y_1'(t)] + [v'(t)y_1'(t) + v(t)y_1''(t)]$$

Substituting the above formulas into Eq.(6.5) we obtain:

$$\{[v''(t)y_1(t)+v'(t)y_1'(t)]+[v'(t)y_1'(t)+v(t)y_1''(t)]\}+p(t)[v'(t)y_1(t)+v(t)y_1'(t)]+q(t)y_1(t)=0.$$

Or equivalently,

$$y_1(t)v'' + [2y_1'(t) + p(t)y_1(t)]v' + [y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)]v = 0.$$

Since $y_1(t)$ is a solution to Eq.(6.5), we derive $y_1''(t) + p(t)y_1'(t) + q(t)y_1(t) = 0$. Thus

$$y_1(t)v'' + [2y_1'(t) + p(t)y_1(t)]v' = 0. \tag{6.6}$$

Notice that Eq.(6.6) is actually a **first-order equation** for function v'. Hence it is possible to write the formula for $v(t)$, but this formula is ugly. In practice we often work on the specific $v(t)$ and derive the final formula.
But, I will show you how to derive $v(t)$ in general case:
For Eq.(6.6), we divide $y_1(t)$ both sides to obtain the **first order linear homogeneous equation**:

$$v'' + \left[2\frac{y_1'(t)}{y_1(t)} + p(t)\right]v' = 0.$$

The solution is obviously given by:

$$v' = C\exp\left[-\int\left(2\frac{y_1'(t)}{y_1(t)} + p(t)\right)\mathrm{d}t\right] \text{ for constant } C$$

It follows that

$$v = C\int\left\{\exp\left[-\int\left(2\frac{y_1'(t)}{y_1(t)} + p(t)\right)\mathrm{d}t\right]\right\}\mathrm{d}t \text{ for constant } C$$

Hence the second solution is given by:

$$y = Cy_1(t)\int\left\{\exp\left[-\int\left(2\frac{y_1'(t)}{y_1(t)} + p(t)\right)\mathrm{d}t\right]\right\}\mathrm{d}t \text{ for constant } C$$

■

Why this method is called *the reduction of order*? Because the *critical* step is to compute the solution to a **first order ODE** for v' instead of the original **second order ODE** for y. We show an example of using reduction of order method:

■ **Example 6.1** Given that $y_1 = t^{1/4}e^{2\sqrt{t}}$ is a solution of

$$t^2y'' - (t - 0.1875)y = 0, \qquad t > 0, \tag{6.7}$$

find a fundamental set of solutions.
We set $y = v(t)y_1(t)$, which follows that

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

$$\begin{aligned} y'' &= [v''(t)y_1(t) + v'(t)y_1'(t)] + [v'(t)y_1'(t) + v(t)y_1''(t)] \\ &= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t). \end{aligned}$$

And we put the formulas above into Eq.(6.7) to obtain:

$$t^2[v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)] - (t - 0.1875)v(t)y_1(t) = 0.$$

Or equivalently,

$$t^2y_1(t)v'' + 2t^2y_1'(t)v' + [t^2y_1''(t) - (t - 0.1875)y_1(t)]v(t) = 0.$$

Since $y_1(t)$ is a solution to Eq.(6.7), we have $t^2y_1''(t) - (t - 0.1875)y_1(t) = 0$. Thus

$$t^2y_1(t)v'' + 2t^2y_1'(t)v' = 0. \implies y_1(t)v'' + 2y_1'(t)v' = 0.$$

Substituting $y_1 = t^{1/4}e^{2\sqrt{t}}$ into the above equation we obtain:

$$t^{1/4}e^{2\sqrt{t}}v'' + 2\left[\frac{1}{4}t^{-3/4}e^{2\sqrt{t}} + t^{-1/4}e^{2\sqrt{t}}\right]v' = 0.$$

Since $e^{2\sqrt{t}} \neq 0$, we obtain:

$$t^{1/4}v'' + \left(\frac{1}{2}t^{-3/4} + 2t^{-1/4}\right)v' = 0. \implies v'' + \left(\frac{1}{2}t^{-1} + 2t^{-1/2}\right)v' = 0.$$

Solving this first order linear homogeneous equation for function v' we derive:

$$\begin{aligned} v' &= C\exp\left[-\int\left(\frac{1}{2}t^{-1} + 2t^{-1/2}\right)\mathrm{d}t\right] = C\exp\left[-\left(\frac{1}{2}\ln|t| + 4t^{1/2}\right)\mathrm{d}t\right] \\ &= Ct^{-1/2}\exp(-4t^{1/2}) \end{aligned}$$

It follows that

$$v = C\int\left[t^{-1/2}\exp(-4t^{1/2})\right]\mathrm{d}t = 2C\int\exp(-4t^{1/2})\,\mathrm{d}t^{1/2} = -\frac{C}{2}\exp(-4t^{1/2}) + D$$

In other words, our finally v is given by:

$$v = c_1\exp(-4t^{1/2}) + c_2$$

It follows that

$$y = v(t)y_1(t) = c_1 t^{1/4}\exp(-2t^{1/2}) + c_2 t^{1/4}\exp(2t^{1/2})$$

Hence our second solution is

$$y_2 = t^{1/4}\exp(-2t^{1/2})$$

And the Wronskian is given by:

$$W(y_1,y_2)(t) = \begin{vmatrix} t^{1/4}\exp(-2t^{1/2}) & t^{1/4}\exp(2t^{1/2}) \\ \exp(-2t^{1/2})\left(\frac{1}{4}t^{-3/4} - t^{-1/4}\right) & \exp(2t^{1/2})\left(\frac{1}{4}t^{-3/4} + t^{-1/4}\right) \end{vmatrix} = 2 \neq 0.$$

Hence the fundamental set of solution to Eq.(6.7) is

$$\{y_1 = t^{1/4}\exp(2t^{1/2}), \quad y_2 = t^{1/4}\exp(-2t^{1/2})\}$$

Thus our general solution is

$$y = c_1 y_1 + c_2 y_2 = t^{1/4}\left[c_1 \exp(2t^{1/2}) + c_2 \exp(-2t^{1/2})\right]$$

■

6.4 Sketch to solve nonhomogeneous equations

We now return to the *nonhomogeneous* equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{6.8}$$

where p, q and g are given continuous functions on open interval I.
How to find its general solution? Actually we need to find its corresponding **homogeneous** equation:

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{6.9}$$

The solution to (6.9) provides a basis for constructing the general solution to (6.8):

Theorem 6.1 Suppose $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions to the homogeneous ODE:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

And $y_0(t)$ is a **particular solution** to the nonhomogeneous ODE:

$$L[y] = y'' + p(t)y' + q(t)y = g(t).$$

Then the general solution to the nonhomogeneous ODE is given by:

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_0(t),$$

where c_1, c_2 are arbitrary constant.

Proof. Suppose $\phi(t)$ is arbitrary solution to Eq.(6.8), then we have:

$$L[\phi] = L[y_0] = g(t)$$

Hence we have:

$$L[\phi] - L[y_0] = L[\phi - y_0] = 0$$

And any solution to the homogeneous equation is a linear combination of y_1 and y_2.
Hence there exists c_1 and c_2 s.t.

$$\phi - y_0 = c_1y_1 + c_2y_2 \implies \phi = c_1y_1 + c_2y_2 + y_0$$

Thus the proof is complete. ∎

The theorem(6.1) states that to solve the nonhomogeneous equation(6.8), we need to do three things:

- Find the *general solution* $c_1y_1 + c_2y_2$ to the *corresponding homogeneous equation*.
- Find **one single** solution $Y(t)$ to the *nonhomogeneous equation*, which is often referred as **a particular solution**.
- Sum the functions found in steps 1 and 2.

We have discussed how to find $c_1y_1 + c_2y_2$, but how to find a **particular solution** in step 2? There are two methods known as *the method of undetermined coefficients* and *the method of variation of parameters*. We will discuss the former one now:

6.5 *Method of undetermined coefficients*

This method requires us to **guess** about the *forms of the particular solution* $Y(t)$, but with the **coefficients undetermined**. And then we substitute this expression into Eq.(6.8). Then we can determine the coefficients to satisfy the Eq.(6.8). The key idea is how do we guess the forms of particular solution? We will introduce some forms corresponding to very special 2nd order ODEs.
Firstly, let's discuss five examples to give you some intuition on this method:

■ **Example 6.2** Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} \tag{6.10}$$

We guess our particular solution $Y(t)$ is some multiple of e^{2t}:

$$Y(t) = Ae^{2t} \text{ where } A \text{ needs to be determined}$$

It follows that

$$Y'(t) = 2Ae^{2t}, \qquad Y''(t) = 4Ae^{2t}$$

After substituting for y, y' and y'' in Eq.(6.10) we obtain:

$$(4A - 6A - 4A)e^{2t} = 3e^{2t} \implies A = -\frac{1}{2}$$

Hence our particular solution is

$$Y(t) = -\frac{1}{2}e^{2t}.$$

■

However, our guess for this kind of ODE ($g(t)$ is exponential) is not always correct. Let me raise a counterexample:

■ **Example 6.3** Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t} \tag{6.11}$$

- If we guess our particular solution $Y(t)$ is some multiple of e^{-t}, i.e. $Y(t) = Ae^{-t}$, then you can verify that after substituting for y, y' and y'' in Eq.(6.11) we obtain:

  $$(A + 3A - 4A)e^{-t} = 2e^{-t} \implies 2e^{-t} = 0$$

  which is a contradiction. Hence there is no particular solution in form of Ae^{-t}.

Why our guess is wrong? Notice that Ae^{-t} is the solution to its corresponding homogeneous equation, so it is impossible for this term to be the solution to the inhomogeneous equation. How do we handle this case?

- We guess our particular solution $Y(t)$ is some of multiple of te^{-t}, i.e.

  $$Y(t) = Ate^{-t}.$$

It follows that

$$Y'(t) = A(1-t)e^{-t}, \qquad Y''(t) = A(t-2)e^{-t}$$

After substituting for y, y' and y'' in Eq.(6.11) we obtain:

$$A(t-2-3+3t-4t)e^{-t} = 2e^{-t} \implies A = -\frac{2}{5}$$

Hence our particular solution is

$$Y(t) = -\frac{2}{5}te^{-t}$$

■

■ **Example 6.4** Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t \tag{6.12}$$

- If we guess our particular solution $Y(t)$ is some multiple of $\sin t$, i.e. $Y(t) = A\sin t$, then you can verify that after substituting for y, y' and y'' in Eq.(6.12) we obtain:

$$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t \implies \begin{cases} -A - 4A = 2 \\ -3A = 0 \end{cases}$$

which is a contradiction. Hence there is no particular solution in form of $A\sin t$.
Why our guess is wrong? We notice that there appears $\cos t$ during our deviation. Hence we should do a little modification for our guess:

- We guess our particular solution $Y(t)$ is linear combination of $\sin t$ and $\cos t$, i.e.

$$Y(t) = A\sin t + B\cos t$$

And it follows that

$$Y'(t) = A\cos t - B\sin t, \qquad Y''(t) = -A\sin t - B\cos t$$

After substituting for y, y' and y'' in Eq.(6.12) we obtain:

$$(-A\sin t - B\cos t) + (-3A\cos t + 3B\sin t) + (-4A\sin t - 4B\cos t) = 2\sin t$$

Or equivalently,

$$(-A+3B-4A)\sin t + (-B-3A-4B)\cos t = 2\sin t \implies \begin{cases} -5A + 3B = 2 \\ -5B - 3A = 0 \end{cases} \implies \begin{cases} A = -\frac{5}{17} \\ B = \frac{3}{17} \end{cases}$$

Hence our particular solution is:

$$Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

■

What information we can summarize from the above examples?

- If nonhomogeneous term $g(t)$ is $e^{\alpha t}$,
  - In most cases ($e^{\alpha t}$ is not a solution to its corresponding homogeneous ODE), we should guess $Y(t)$ is $Ae^{\alpha t}$. [like Example(6.2)]
  - If $e^{\alpha t}$ is also a solution to its corresponding homogeneous ODE, then we should guess $Y(t)$ is $Ate^{\alpha t}$. [like Example(6.3)]
- If $g(t)$ is $\sin\beta t$ or $\cos\beta t$, then we should guess $Y(t)$ is a linear combination of $\sin\beta t$ **and** $\cos\beta t$. [like Example(6.4)]

But what if $g(t)$ is $e^{\alpha t}\sin\beta t$ or $e^{\alpha t}\cos\beta t$? We just take the product of the corresponding types of functions, i.e.

$$Y(t) = Ae^{\alpha t}(B\sin\beta t + C\cos\beta t) \text{ or } Y(t) = Ate^{\alpha t}(B\sin\beta t + C\cos\beta t)$$

Later we will show the general idea of how to guess our $Y(t)$.

■ **Example 6.5** Find a particular solution of

$$y'' - 3y' - 4y = -8e^t\cos 2t. \tag{6.13}$$

We guess our particular solution $Y(t)$ is the product of e^t and *a linear combination of* $\cos 2t$ *and* $\sin 2t$, i.e.

$$Y(t) = Ae^t\cos 2t + Be^t\sin 2t$$

It follows that

$$\begin{aligned} Y'(t) &= [A\cos 2t - 2A\sin 2t]e^t + [B\sin 2t + 2B\cos 2t]e^t \\ &= (A+2B)e^t\cos 2t + (-2A+B)e^t\sin 2t \end{aligned}$$

and

$$\begin{aligned} Y''(t) &= [(A+2B)\cos 2t - 2(A+2B)\sin 2t]e^t + [(-2A+B)\sin 2t + 2(-2A+B)\cos 2t]e^t \\ &= (-3A+4B)e^t\cos 2t + (-4A-3B)e^t\sin 2t \end{aligned}$$

After substituing for y, y' and y'' in Eq.(6.13) we obtain:

$$\begin{aligned} &e^t\cos 2t[(-3A+4B) - 3(A+2B) - 4A] \\ &+e^t\sin 2t[(-4A-3B) - 3(-2A+B) - 4B] = -8e^t\cos 2t \end{aligned}$$

Hence we derive:

$$\begin{cases} -10A - 2B = -8 \\ \;\;2A - 10B = 0 \end{cases} \Longrightarrow \begin{cases} A = \dfrac{10}{13} \\ B = \dfrac{2}{13} \end{cases}$$

Hence our particular solution is:

$$Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t.$$

■

Now we suppose that $g(t)$ is a sum of two terms, $g(t) = g_1(t) + g_2(t)$. And we know that $Y_1(t)$ and $Y_2(t)$ are two solutions of the equations

$$ay'' + by' + cy = g_1(t) \tag{6.14}$$
$$ay'' + by' + cy = g_2(t) \tag{6.15}$$

Then how to find the particular solution for the equation

$$ay'' + by' + cy = g(t) = g_1(t) + g_2(t)? \tag{6.16}$$

Since $Y_1(t)$ and $Y_2(t)$ are two solutions of the Eq.(6.14) and Eq.(6.15), we have

$$aY_1'' + bY_1' + cY_1 = g_1(t) \tag{6.17}$$
$$aY_2'' + bY_2' + cY_2 = g_2(t) \tag{6.18}$$

Then Eq.(6.17)+Eq.(6.18) follows that

$$\begin{aligned}[aY_1'' + bY_1' + cY_1] + [aY_2'' + bY_2' + cY_2] &= a[Y_1'' + Y_2''] + b[Y_1' + Y_2'] + c[Y_1 + Y_2] \\ &= a[Y_1 + Y_2]'' + b[Y_1 + Y_2]' + c[Y_1 + Y_2] \\ &= g_1(t) + g_2(t) = g(t).\end{aligned}$$

Hence we derive that $Y_1 + Y_2$ is a particular solution to Eq.(6.16). The following example shows this procedure.

■ **Example 6.6** Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2e^{-t} + 2\sin t - 8e^t \cos 2t. \tag{6.19}$$

By splitting up the RHS of Eq.(6.19) we obtain the four equations:

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2e^{-t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^t \cos 2t$$

Through the Example(6.2) to Example(6.5) we can find the particular solution of these equations.
Thus a particular solution to Eq.(6.19) is a sum of them, i.e.

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{2}{5}te^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

■

■ **Solution 6.3** Now we summarize how to find the general solution to the nonhomogeneous equation:

$$ay'' + by' + cy = g(t) \tag{6.20}$$

1. Find the general solution $y_c(t)$ of the corresponding homogeneous equation.
   - The specific way is to use the characteristic equation $ar^2 + br + c = 0$. Then

discuss it in three cases:

$$\begin{cases} \Delta > 0: & y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \\ \Delta = 0: & y(t) = c_1 e^{rt} + c_2 t e^{rt}. \\ \Delta < 0: & y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t). \end{cases}$$

2. You should make sure that our $g(t)$ only belong to the class of functions discussed below:

   $e^{\alpha t}$ (exponential) $\sin \alpha t$ or $\cos \alpha t$ (triangle) $P_n(t)$ (polynomial)

   or the **sums** or the **products** of such functions above. Otherwise we **cannot** solve this ODE now, we have to use *variation of parameters* discuessed in the next section!
   - Caution: Note that polynomials doesn't contain *fraction function*, such as t^{-1}.
3. Then we need to find the **particular solution** $Y(t)$ to Eq.(6.20) using *method of undetermined coefficients*. But how to guess the forms of our $Y(t)$? You can check the table below as reference. (*try to memorize it, you can check the example discussed before to understand this table.*)

   Note: Here α is a "*root*" refers to α is a "*root*" of the characterstic equation $ar^2 + br + c = 0$.

$g(t)$	Y(t)	The value for s
$ke^{\alpha t}$	$At^s e^{\alpha t}$	$s = \begin{cases} 0, \alpha \text{ is not a root.} \\ 1, \alpha = r_1 \neq r_2 \\ 2, \alpha = r_1 = r_2 \end{cases}$
$P_n(t)e^{\alpha t}$	$Q_n(t)t^s e^{\alpha t}$	**Same as above**
$\begin{cases} P_n e^{\alpha t} \sin \beta t \\ P_n e^{\alpha t} \cos \beta t \end{cases}$	$\begin{cases} [Q_n(t) \cos \beta t \\ +R_n(t) \sin \beta t] t^s e^{\alpha t} \end{cases}$	$s = \begin{cases} 0, \alpha + i\beta \text{ is not a root.} \\ 1, \alpha + i\beta = r_1, \alpha - i\beta = r_2 \end{cases}$

   After guessing the fomrs of $Y(t)$, we derive Y' and Y'', then plug them into our Eq.(6.20) to derive the coefficients. Thus we obtain our particular solution $Y(t)$.
4. Finally, then general solution to Eq.(6.20) is given by:

$$y = y_c(t) + Y(t).$$

■

$y_0(t) = C_1(t)Y_1(t) + C_2(t)Y_2(t)$ Question: How to choose $C_1(t)$ and $C_2(t)$? Let $y = y_0(t)$, then we have

$$\begin{aligned} y' &= C_1'Y_1 + C_1Y_1' + C_2'Y_2 + C_2Y_2' \\ &= [C_1'Y_1 + C_2'Y_2] + [C_1Y_1' + C_2Y_2'] \end{aligned}$$

We let $C_1'Y_1 + C_2'Y_2 = 0$ (Condition One), then

$$y' = C_1Y_1' + C_2Y_2'$$

It follows that

$$\begin{aligned} y'' &= [C_1'Y_1' + C_1Y_1''] + [C_2'Y_2' + C_2Y_2''] \\ &= [C_1'Y_1' + C_2'Y_2'] + [C_1Y_1'' + C_2Y_2''] \end{aligned}$$

Then we have

$$\begin{aligned} g &= y'' + py' + qy \\ &= [C_1'Y_1' + C_2'Y_2'] + [C_1Y_1'' + C_2Y_2''] + p[C_1Y_1' + C_2Y_2'] + q[C_1Y_1 + C_2Y_2]] \\ &= [C_1'Y_1' + C_2'Y_2'] + C_1[Y_1'' + pY_1' + qY_1] + C_2[Y_2'' + pY_2' + qY_2] \\ &= C_1'Y_1' + C_2'Y_2' \end{aligned}$$

Second Condition: $g = C_1'Y_1' + C_2'Y_2'$.

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix} \text{ matrix form}$$

Must: Determinant $W(Y_1, Y_2)(t) \neq 0$, there exists unqiue pair of (C_1', C_2'):

$$C_1' = \frac{\begin{vmatrix} 0 & Y_2 \\ g & Y_2' \end{vmatrix}}{W(Y_1, Y_2)(t)} = -\frac{Y_2(t)g(t)}{W(Y_1, Y_2)(t)}$$

and

$$C_2' = \frac{\begin{vmatrix} Y_1 & 0 \\ Y_1' & g \end{vmatrix}}{W(Y_1, Y_2)(t)} = \frac{Y_1(t)g(t)}{W(Y_1, Y_2)(t)}$$

Hence we obtain:

$$C_1(t) = -\int \frac{Y_2(t)g(t)}{W(Y_1, Y_2)(t)}\,\mathrm{d}t \qquad C_2(t) = \int \frac{Y_1(t)g(t)}{W(Y_1, Y_2)(t)}\,\mathrm{d}t$$

Theorem 6.2 Assume that p, q, g are c.n.t. on I, and $\{Y_1, Y_2\}$ ae a fundamental set of solutions to homogeneous Eq, then a particular solution to (1) is

$$y_0(t) = -\int \frac{Y_2(t)g(t)}{W(Y_1, Y_2)(t)}\,\mathrm{d}t Y_1(t) + \int \frac{Y_1(t)g(t)}{W(Y_1, Y_2)(t)}\,\mathrm{d}t Y_2(t)$$

The general solution is given by

$$y(t) = C_1Y_1 + C_2Y_2 + y_0(t)$$

for arbitrary constants C_1, C_2.

7 — Week7

7.1 Method of variation of parameters

Like the method of undetermined coefficients, this method is also designed to find the particular solution of a **nonhomogeneous** 2nd order ODE.

- Why do we learn this method?
  Because one limitation of *method of undetermined coefficients* is that it cannot handle general $g(t)$. But this method could.
- In which case does it work?
  The *method of variation of parameters* works for the general 2nd order linear ODE when the two conditions are satisfied:
  1. We have known $\{Y_1, Y_2\}$ is a fundamental set of solutions to its corresponding **homogeneous** equations.
  2. The integral in the expression of the particular solution can be solved in terms of *elementary functions*.

7.1.1 How to use this method to solve our ODE?

Given general linear inhomogeneous ODE and its corresponding homogeneous ODE:

$$y'' + p(t)y' + q(t)y = g(t) \tag{7.1}$$
$$y'' + p(t)y' + q(t)y = 0 \tag{7.2}$$

Suppose $\{Y_1, Y_2\}$ is a fundamental set of solutions to Eq.(7.2), i.e.

$$\begin{cases} Y_1, Y_2 \text{ solves for Eq.(7.2)} \\ W(Y_1, Y_2)(t) \neq 0 \end{cases}$$

■ **Solution 7.1** How to solve Eq.(7.1)? We guess our particular solution to be:

$$y_0(t) = C_1(t)Y_1(t) + C_2(t)Y_2(t)$$

Question: How to choose $C_1(t)$ and $C_2(t)$?
Let $y = y_0(t)$, then we have

$$\begin{aligned} y' &= C_1'Y_1 + C_1Y_1' + C_2'Y_2 + C_2Y_2' \\ &= [C_1'Y_1 + C_2'Y_2] + [C_1Y_1' + C_2Y_2'] \end{aligned}$$

If we set

$$C_1'Y_1 + C_2'Y_2 = 0 \tag{7.3}$$

then it follows that

$$y' = C_1Y_1' + C_2Y_2'$$

and

$$\begin{aligned} y'' &= [C_1'Y_1' + C_1Y_1''] + [C_2'Y_2' + C_2Y_2''] \\ &= [C_1'Y_1' + C_2'Y_2'] + [C_1Y_1'' + C_2Y_2''] \end{aligned}$$

Substituting y, y' and y'' in Eq.(7.1) we derive:

$$\begin{aligned} g(t) &= y'' + py' + qy \\ &= [C_1'Y_1' + C_2'Y_2'] + [C_1Y_1'' + C_2Y_2''] + p[C_1Y_1' + C_2Y_2'] + q[C_1Y_1 + C_2Y_2]] \\ &= [C_1'Y_1' + C_2'Y_2'] + C_1[Y_1'' + pY_1' + qY_1] + C_2[Y_2'' + pY_2' + qY_2] \\ &= C_1'Y_1' + C_2'Y_2' \end{aligned}$$

Hence we only need to solve the equation:

$$g = C_1'Y_1' + C_2'Y_2' \tag{7.4}$$

Combining Eq.(7.3) and Eq.(7.4) into compact matrix form:

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix} \tag{7.5}$$

So we only need to solve Eq.(7.5).
In order to solve it, we let the determinant $W(Y_1,Y_2)(t) \neq 0$, then there exists unqiue pair of (C_1', C_2') s.t.

$$C_1' = \frac{\begin{vmatrix} 0 & Y_2 \\ g & Y_2' \end{vmatrix}}{W(Y_1,Y_2)(t)} = -\frac{Y_2(t)g(t)}{W(Y_1,Y_2)(t)} \qquad C_2' = \frac{\begin{vmatrix} Y_1 & 0 \\ Y_1' & g \end{vmatrix}}{W(Y_1,Y_2)(t)} = \frac{Y_1(t)g(t)}{W(Y_1,Y_2)(t)}$$

By integrating for C_1' and C_2' we obtain:

$$C_1(t) = -\int \frac{Y_2(t)g(t)}{W(Y_1,Y_2)(t)}\,\mathrm{d}t \qquad C_2(t) = \int \frac{Y_1(t)g(t)}{W(Y_1,Y_2)(t)}\,\mathrm{d}t$$

Hence our particular solution is given by:

$$Y(t) = C_1(t)Y_1(t) + C_2(t)Y_2(t).$$

■

The part discussed above could be summarize into one theorem, which can be used directly in exam:

Theorem 7.1 Assume that p,q,g are c.n.t. on an open interval I, and $\{Y_1,Y_2\}$ ae a fundamental set of solutions to homogeneous Eq.(7.2), then a **particular solution** to Eq.(7.1) is

$$y_0(t) = -\left[\int \frac{Y_2(t)g(t)}{W(Y_1,Y_2)(t)}\,\mathrm{d}t\right]Y_1(t) + \left[\int \frac{Y_1(t)g(t)}{W(Y_1,Y_2)(t)}\,\mathrm{d}t\right]Y_2(t)$$

The general solution is given by

$$y(t) = \underbrace{C_1Y_1 + C_2Y_2}_{\text{solution to homogeneous ODE}} + \underbrace{y_0(t)}_{\text{particular solution}}$$

for arbitrary constants C_1,C_2.

Here we show two examples on how to sovle general inhomogeneous 2nd order linear ODE:

■ **Example 7.1** Find general solution to the ODE:

$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0. \tag{7.6}$$

- The characteristic equation is:

$$r^2 + 4r + 4 = 0 \implies r_1 = r_2 = -2.$$

  And the Wronskian is given by:

$$W(e^{-2t},te^{-2t})(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}\begin{vmatrix} 1 & t \\ -2 & 1-2t \end{vmatrix} = e^{-4t} \neq 0$$

  Hence the solution for the homogeneous part is:

$$y = C_1e^{-2t} + C_2te^{-2t}$$

- We guess our particular solution to be

$$y_0(t) = C_1(t)e^{-2t} + C_2(t)te^{-2t}$$

Then by Theorem(7.1), we derive:

$$C_1(t) = -\int \frac{Y_2 g}{W(Y_1,Y_2)}\,\mathrm{d}t = -\int \frac{te^{-2t}\times t^{-2}e^{-2t}}{e^{-4t}}\,\mathrm{d}t = -\int \frac{1}{t}\,\mathrm{d}t = -\ln t$$
$$C_2(t) = \int \frac{Y_1 g}{W(Y_1,Y_2)}\,\mathrm{d}t = \int \frac{e^{-2t}\times t^{-2}e^{-2t}}{e^{-4t}}\,\mathrm{d}t = \int \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{t}$$

Hence our particular solution is given by:

$$y_0(t) = -\ln t e^{-2t} - e^{-2t}$$

Combining the two parts, the general solution is

$$y = C_1 e^{-2t} + C_2 t e^{-2t} - \ln t e^{-2t}.$$

■

■ **Example 7.2** Find the general solution to the ODE:

$$t^2 y'' + 7ty' + 5y = 3t, \quad t > 0, \quad y_1(t) = t^{-1} \tag{7.7}$$

- Firstly, we use reduction of order to find another solution to the homogeneous part:

$$y(t) = v(t)y_1(t) = t^{-1}v(t)$$

It follows that:

$$y' = -t^{-2}v(t) + t^{-1}v'(t)$$

and

$$\begin{aligned} y'' &= 2[t^{-3}v(t) - t^{-2}v'(t)] + [-t^{-2}v'(t) + t^{-1}v''(t)] \\ &= 2t^{-3}v(t) - 2t^{-2}v'(t) + t^{-1}v''(t). \end{aligned}$$

Substituting y, y' and y'' for the homogeneous ODE to obtain:

$$[2t^{-1}v(t) - 2v'(t) + tv''(t)] + 7[-t^{-1}v(t) + v'(t)] + 5t^{-1}v(t) = 0.$$

Or equivalently,

$$5v'(t) + tv''(t) = 0 \implies v'(t) + \frac{t}{5}v''(t) = 0$$

Hence we derive:

$$v'(t) = \exp\left[-\int \frac{t}{5}\,\mathrm{d}t\right] = Ct^{-5}. \implies v(t) = C_1 t^{-4} + C_2$$

Hence $y(t) = t^{-1}v(t) = C_2t^{-1} + C_1t^{-5}$.
Thus another solution to the homogeneous part is

$$y_2(t) = t^{-5}.$$

And the Wronskian is given by:

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{vmatrix} = -4t^{-7} \neq 0$$

Hence $\{t^{-1}, t^{-5}\}$ forms fundamental set of solutions to the homogeneous part, the solution to which is

$$y = C_1t^{-1} + C_2t^{-5}.$$

- And we guess our particular solutio to be

$$y_0(t) = C_1(t)t^{-1} + C_2(t)t^{-5}.$$

Then by Theorem(7.1), we derive: (Here you should think why $g = 3t^{-1}$.)

$$C_1(t) = -\int \frac{y_2 g}{W(y_1, y_2)}\,\mathrm{d}t = -\int \frac{t^{-5} \times 3t^{-1}}{-4t^{-7}}\,\mathrm{d}t = -\int -\frac{3}{4}t\,\mathrm{d}t = \frac{3}{8}t^2$$
$$C_2(t) = \int \frac{y_1 g}{W(y_1, y_2)}\,\mathrm{d}t = \int \frac{t^{-1} \times 3t^{-1}}{-4t^{-7}}\,\mathrm{d}t = \int -\frac{3}{4}t^5\,\mathrm{d}t = -\frac{1}{8}t^6$$

Hence our particular solution is given by:

$$y_0(t) = \frac{3}{8}t \cdot t^{-1} - \frac{1}{8}t^6 \cdot t^{-5} = \frac{1}{4}t.$$

Combining the two parts, the general solution is

$$y = C_1t^{-1} + C_2t^{-5} + \frac{1}{4}t.$$

■

7.2 Application to Vibrations

Why do we study the second order ODE? Because many phenomenon in our nature could be expressed as this kind of ODE. Let's raise a specific example about this:

■ **Example 7.3** We attach a mass into a spring as Figure(7.1) shown below:

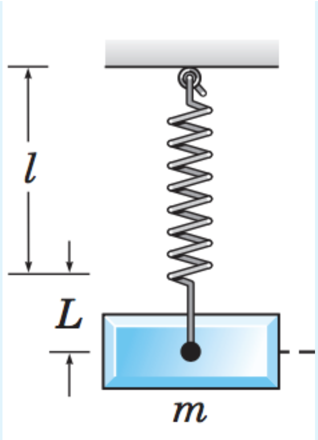


Figure 7.1: A spring-mass system (still condition)

When this system is still, we have:

$$mg = kL.$$

But when we let it oscillate a little bit, what will happen? We build a mathematical model to describe it:

Gravity	$G = mg$
Position	$\mu(t)$
Spring force	$F_s = -k[L + \mu(t)]$
Air resistence	$F_d = -r\mu'(t)$
External force	$F(t)$
Total Force	$G + F_s + F_d + F(t)$

We apply the *Newton's second law* to obtain:

$$\begin{aligned} m\mu''(t) &= mg - k[L + \mu(t)] - r\mu'(t) + F(t) \\ &= -k\mu(t) - r\mu'(t) + F(t) \end{aligned}$$

Thus we obtain our second order IVP:

$$\begin{cases} m\mu''(t) = -k\mu(t) - r\mu'(t) + F(t) \\ \mu(0) = \mu_0, \quad \mu'(0) = \mu_0' \end{cases}$$

■

8 — Week8

The fundamental theorems and statements towards high-order linear ODEs will be introduced in this lecture.

Definition 8.1 — Order n differential equations. A **differential equation** of order n is an equation

$$\frac{d^n y}{dt^n} = F\left(y, \frac{dy}{dt}, \cdots, \frac{d^{n-1}y}{dt^{n-1}}\right) \tag{8.1}$$

where F is a **differentiable** function defined in a domain U of a $n+1$ dimension space.

There are two kind of **linear** ODEs:

- Nonhomogeneous ODEs:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{8.2}$$

- Homogeneous ODEs:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0 \tag{8.3}$$

■ **Example 8.1** Unlike first order linear ODE, the solutions to high order ODE may intersect with each other. Here is an example:
For ODE $y'' = -y$, the two fundamental solutions are $y_1 = \sin t$, $y_2 = \cos t$. As we can see in Figure(8.1), **the solutions of a second order ODE may intersect**.

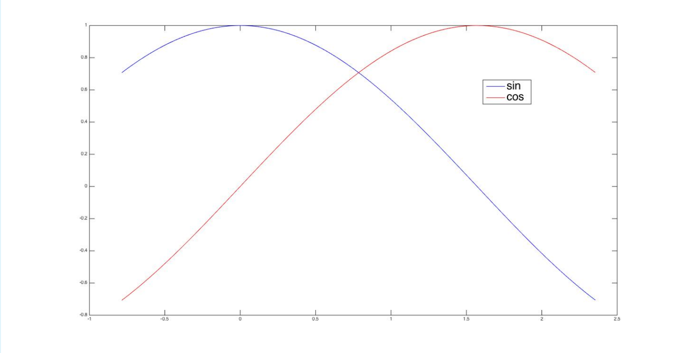


Figure 8.1: The graph of two solutions of a second-order ODE

Recall that due to the uniqueness theorem of first order ODE, two distinct solutions of first order ODE will not intersect. Hence this theorem don't apply for the high order ODE. Our question is that **what initial conditions should be given in order to determine the uniqueness of the high order ODE**?

Definition 8.2 — Existence and Uniqueness Theorem. Let $(y_0, y_0^{(1)}, \dots, y_0^{(n-1)})$ be a point such that Eq.(8.2) is satisfied, i.e.

$$y_0^{(n-1)} + p_1(t)y_0^{(n-1)} + \cdots + p_{n-1}(t)y_0^{(1)} + p_n(t)y_0 = g(t)$$

If $p_1, p_2, \dots, p_n$ and g are continuous on open interval I, and t_0 is arbitrary point on I. Then the solution ϕ to Eq.(8.2) with initial condition

$$\phi(t_0) = y_0, \quad \phi'(t_0) = y_0^{(1)}, \quad \cdots, \quad \phi^{(n-1)}(t_0) = y_0^{(n-1)} \tag{8.4}$$

exists and is unique.

In other words, this theorem guarantees that if any two solutions have the same initial condition, then these two solutions will be the same on the interval I.

■ **Example 8.2** Look at Example(8.1) again, at $t = \frac{\pi}{4}$, the solution y_1 satisfy

$$y_1\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad y_1'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

The solution y_2 satisfy

$$y_2\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad y_2'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

So these initial conditions are distinct, so it is not surpring that graphs of the solutions intersect without coinciding.

Exercise 8.1 Suppose we know the Eq.(8.2) has solutions $y_1 = t$ and $y_2 = \sin t$, determine the range of order n of the equation.

Solution. Notice that y_1 and y_2 have the same derivatives of orders 0, 1 and 2 at point $t = 0$. If they satisfy the same **third-order** equation, they would be the same on the open interval I due to the uniqueness theorem. So an equation of order $n \geq 4$ satisfies both funtion. For example, one equation is $y^{(n)} + y^{(n-2)} = 0, \ \ n \geq 4$. Hence $n \geq 4$. ■

Exercise 8.2 Can the graph of two solutions of the equation $y'' + p(t)y' + q(t)y = 0$ have the form depicted in Figure(8.2)?

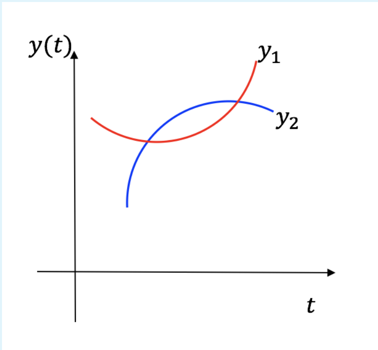


Figure 8.2: An impossible configuration of graphs

Proof. No. We can multiple solution y_1 with constant c to make cy_1 and y_2 have the same initial condition: (the reason why we can do this is shown below)
We construct $g_1 = \ln(y_1)$ and $g_2 = \ln(y_2)$. From the Figure(8.2), we assume $y_1(a) = y_2(a), \ \ y_1(b) = y_2(b)$. It follows that

$$g_1(a) = g_2(a), \ \ g_1(b) = g_2(b)$$

Due to the **Roll's theorem**(Calculus I), there exists $d \in (a,b)$ such that $g_1'(d) = g_2'(d)$. It follows that there exists d s.t.

$$\frac{y_1'(d)}{y_1(d)} = \frac{y_2'(d)}{y_2(d)}$$

So it is always possible to multiply c to make

$$cy_1'(d) = y_2'(d) \quad cy_1(d) = y_2(d) \text{ Shown in Fiugre(8.3)}$$

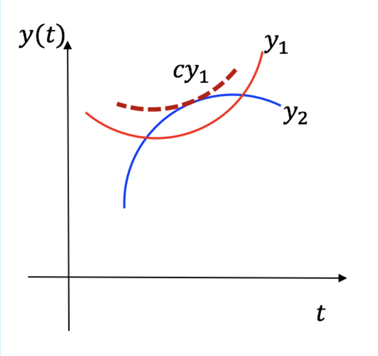


Figure 8.3: An impossible configuration of graphs

But functions cy_1 and y_2 don't coincide, which is a contradiction! ■

Definition 8.3 — Fundamental set of solutions.
Suppose $\{y_1,\dots,y_n\}$ are solutions to Eq.(8.3). And $\{y_1,\dots,y_n\}$ is said to form a **fundamental set of solutions** if *every solution to Eq.(8.3) could be expressed as a linear combination of solutions* $y_1,\dots,y_n$ *uniquely.*

Our question is how to verify a set of solutions is a fundamental set of solutions? Firstly we define the Wronskian for high order ODE:

Definition 8.4 — Wronskian. The Wronskian for a set of solutions $\{y_1,\dots,y_n\}$ is a special determinant:

$$W(y_1,y_2,\dots,y_n)(t)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem 8.1 — Test for fundamental set of solutions.
The funtions $p_1,\dots,p_n$ are continuous on the open interval I. $\{y_1,y_2,\dots,y_n\}$ forms a fundamental set of solutions for Eq.(8.3) if and only if the Wronskian $W(y_1,y_2,\dots,y_n)(t_0)\neq 0$ for some $t_0\neq 0$.

Proof. Sufficiency. For arbitrary point $t\in I$, any solution y to Eq.(8.3) is uniquely determined if we have defined its initial point:

$$y(t_0)=y_0,\ \ y'(t_0)=y_0',\ \ \cdots,\ \ y^{(n-1)}(t_0)=y_0^{(n-1)}$$

Suppose $\{y_1,y_2,\dots,y_n\}$ forms a fundamental set of solutions for Eq.(8.3), then there exists unique $c_1,c_2,\dots,c_n$ such that

$$c_1y_1+c_2y_2+\cdots+c_ny_n=y$$

It follows that

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\cdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

We write this equation into compact matrix form:

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}$$

Since $\{c_1, c_2, \ldots, c_n\}$ is uniquely defined, the first matrix on the left side must be invertible. Its determinant is nonzero at $t = t_0$. Hence $W(y_1, y_2, \ldots, y_n) \neq 0$.
Necessity. If $W(y_1, y_2, \ldots, y_n)(t_0) \neq 0$, we only need to show that there exists a unique solution $\{c_1, c_2, \ldots, c_n\}$ to the linear system

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\cdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

for some $t_0 \in I$. Since $W(y_1, y_2, \ldots, y_n) \neq 0$ for some $t_0 \in I$, the coefficients $\{c_1, c_2, \ldots, c_n\}$ is uniquely given below:

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}.$$

Hence every solution to Eq.(8.3) could be expressed uniquely as a linear combination of $\{y_1, y_2, \ldots, y_n\}$. ■

We find that we could use Wronskian to describe **fundamental set of solutions**, and moreovre, we could show that the Wronskian is either **zero** for every $t \in I$ or else is **never zero** there.

Theorem 8.2 — Abel's Theorem. For nth high order ODE

$$y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = 0$$

with fundamental set of solutions $y_1, y_2, \ldots, y_n$. The Wronskian formula is given by:

$$W(y_1, y_2, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\,\mathrm{d}t\right] \tag{8.5}$$

Proof. We only need to show that

$$W' + p_1(t)W = 0 \Longleftarrow W' = -p_1(t)W.$$

- By **Leibniz formula for determinants** (check wikipedia for detailed definition), the Wronskian could be expressed as:

$$W(y_1, y_2, \ldots, y_n)(t) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} y_i^{(\sigma(i)-1)}(t)$$

Then we differentiate Wronskian with respect to t:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} W(y_1, y_2, \ldots, y_n)(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} y_i^{(\sigma(i)-1)}(t) \\
&= \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \left(\frac{\mathrm{d}}{\mathrm{d}t} \prod_{i=1}^{n} y_i^{(\sigma(i)-1)}(t) \right) \\
&= \sum_{i=1}^{n} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) y_i^{(\sigma(1)-2)}(t) \prod_{i \in \{1,\ldots,n\}-\{i\}} y_i^{(\sigma(i)-1)}(t) \\
&= \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \\
&\quad + \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}
\end{aligned} \tag{8.6}$$

In other words, the derivative for Wronskian could be calculated by differentiating each row separately.

- Note that the first $n-1$ terms of right side of Eq.(8.6) are all zero since they all contain a pair of identical rows. Hence we derive:

$$\frac{\mathrm{d}}{\mathrm{d}t} W(y_1, y_2, \ldots, y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \tag{8.7}$$

Since every y_i solves the ODE, we have:

$$y_i^{(n)} + p_1(t) y_i^{(n-1)} + \cdots + p_{n-1}(t) y_i' + p_n(t) y_i = 0 \tag{8.8}$$

It follows that

$$y_i^{(n)}+\cdots+p_{n-1}(t)y_i'+p_n(t)y_i=-p_1(t)y_i^{(n-1)}$$

If we add *first row times* $p_n(t)$, *second row times* $p_{n-1}(t)$,..., *the* $(n-1)$*th row times* $p_2(t)$, into the nth row, the value for Eq.(8.7) is invariant:

$$\frac{\mathrm{d}}{\mathrm{d}t}W(y_1,y_2,\dots,y_n)(t)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ -p_1(t)y_1^{(n-1)} & -p_1(t)y_2^{(n-1)} & \cdots & -p_1(t)y_n^{(n-1)} \end{vmatrix}$$

Or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}W(y_1,y_2,\dots,y_n)(t)=-p_1(t)\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}=-p_1(t)W(y_1,y_2,\dots,y_n)(t)$$

■

Now we want to explore the relationship between linear independence and fundamental set of solutions.

Definition 8.5 — Linear independence. The functions $f_1,f_2,\dots,f_n$ is said to be **linearly dependent** on open interval I if there exists $k_1,k_2,\dots,k_n$ which are not all zero s.t.

$$k_1f_1+k_2f_2+\cdots+k_nf_n=0$$

for all $t\in I$. Otherwise the functions $f_1,f_2,\dots,f_n$ are said to be **linearly independent** on I. ■

Theorem 8.3 — Test for fundamental set of solutions.
The funtions $p_1,\dots,p_n$ are continuous on the open interval I. And suppose $y_1,y_2,\dots,y_n$ are solutions to Eq.(8.3). $\{y_1,y_2,\dots,y_n\}$ forms a fundamental set of solutions for Eq.(8.3) if and only if $\{y_1,y_2,\dots,y_n\}$ are linerarly independent on I.

Proof. *Suffiency.* If $\{y_1,y_2,\dots,y_n\}$ forms a fundamental set of solutions for Eq.(8.2), then $W(y_1,y_2,\dots,y_n)(t_0)\neq 0$ for some $t_0\in I$. Due to Abel's theorem, $W(y_1,y_2,\dots,y_n)(t)=0$ for all $t\in I$. Then we consider the equation

$$\alpha_1y_1+\alpha_2y_2+\cdots+\alpha_ny_n=0 \tag{8.9}$$

We differentiate this equation repeatedly to obtain other $(n-1)$ equations:

$$\begin{aligned} \alpha_1y_1'+\alpha_2y_2'+\cdots+\alpha_ny_n'&=0 \\ &\vdots \\ \alpha_1y_1^{(n-1)}+\alpha_2y_2^{(n-1)}+\cdots+\alpha_ny_n^{(n-1)}&=0 \end{aligned} \tag{8.10}$$

We could write Eq.(8.9) and Eq.(8.10) into compact matrix form:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8.11}$$

■

Since $W(y_1, y_2, \ldots, y_n)(t) \neq 0$, the first matrix in Eq.(8) is invertible. Hence the coefficients for Eq.(8.9) should all be zero, which implies that $\{y_1, y_2, \ldots, y_n\}$ are linearly independent.
Necessity. Suppose $y_1, y_2, \ldots, y_n$ are linearly independent on I, we need to show they form a fundamental solution on I. It suffices to show the Wronskian $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for all $t \in I$. Assuming it is not true, and there $\exists t_0 \in I$ such that $W(y_1, y_2, \ldots, y_n)(t_0) = 0$.
In this case the Eq.(8) has a nonzero solution, let's denote it as $\{\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0\}$. Now we construct a linear combination

$$\psi = \alpha_1^0 y_1 + \alpha_2^0 y_2 + \cdots + \alpha_n^0 y_n \tag{8.12}$$

Since we have

$$\begin{aligned} \alpha_1^0 y_1(t_0) + \alpha_2^0 y_2(t_0) + \cdots + \alpha_n^0 y_n(t_0) &= 0 \\ \alpha_1^0 y_1'(t_0) + \alpha_2^0 y_2'(t_0) + \cdots + \alpha_n^0 y_n'(t_0) &= 0 \\ &\vdots \\ \alpha_1^0 y_1^{(n-1)}(t_0) + \alpha_2^0 y_2^{(n-1)}(t_0) + \cdots + \alpha_n^0 y_n^{(n-1)}(t_0) &= 0 \end{aligned}$$

We derive the function ψ satisfies the intial condition:

$$\psi(t_0) = 0, \quad \psi'(t_0) = 0, \quad \ldots, \quad \psi^{(n-1)}(t_0) = 0. \tag{8.13}$$

Moreover, ψ is also a solution to Eq.(8.3). And we find that $y \equiv 0$ is also a solution to Eq.(8.3) which satisfies the initial condition (8.13). By the uniqueness theorem,

$$\psi = \alpha_1^0 y_1 + \alpha_2^0 y_2 + \cdots + \alpha_n^0 y_n = 0.$$

Since $\{\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0\}$ are not all zero, we derive $\{y_1, y_2, \ldots, y_n\}$ are **linearly dependent**, which is a contradiction. Thus the Wronskian is never zero in I, thus $\{y_1, y_2, \ldots, y_n\}$ forms a fundamental set of solutions.

8.1 Application to constant coefficient

We now consider the nth order linear homogeneous differential equation with constant coefficients:

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0, \tag{8.14}$$

where $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$.
If we guess the solution to Eq.(8.14) has the form $y = e^{rt}$, and we plug it into the equation to obtain:

$$e^rt\left[a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n\right] = 0.$$

Since $e^{rt} \neq 0$, we derive:

$$a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0. \tag{8.15}$$

The Eq.(8.15) is called the **characteristic equation**, and the polynomial on the left side is called the **characteristic polynomial**. Now we discuss how to solve the ODE(8.14) by solving the corresponding characteristic equation:

All roots of Eq.(8.15) are real and distinct
In this case we have n distinct solutions $e^{r_1t}, e^{r_2t}, \ldots, e^{r_nt}$. If these functions are linearly independent, the general solution of Eq.(8.14) is:

$$y = c_1e^{r_1}t + c_2e^{r_1t} + \cdots + c_ne^{r_nt}$$

And let's establish the linear independence of $e^{r_1}t, e^{r_2t}, \ldots, e^{r_nt}$ by showing the Wronskian is **nonzero**:

Proof.

$$W(e^{r_1t}, e^{r_2t}, \ldots, e^{r_nt})(t) = \begin{vmatrix} e^{r_1t} & e^{r_2t} & \cdots & e^{r_nt} \\ r_1e^{r_1t} & r_2e^{r_2t} & \cdots & r_ne^{r_nt} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1}e^{r_1t} & r_2^{n-1}e^{r_2t} & \cdots & r_n^{n-1}e^{r_nt} \end{vmatrix}$$

Since $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$ for any matrix $\boldsymbol{A}$, we derive:

$$W(e^{r_1t}, e^{r_2t}, \ldots, e^{r_nt})(t) = \begin{vmatrix} e^{r_1t} & r_1e^{r_1t} & \cdots & r_1^{n-1}e^{r_1t} \\ e^{r_2t} & r_2e^{r_2t} & \cdots & r_2^{n-1}e^{r_2t} \\ \vdots & \vdots & \ddots & \vdots \\ e^{r_nt} & r_ne^{r_nt} & \cdots & r_n^{n-1}e^{r_nt} \end{vmatrix} \tag{8.16}$$

We set the matrix on the right side of Eq.(8.16) to be $\boldsymbol{B}$. In order to show the Wronskian is nonzero, we only need to show $\boldsymbol{B}$ is nonsingular. Notice that

$$\boldsymbol{B\alpha} = \boldsymbol{0} \Longleftrightarrow \alpha_1e^{r_it} + \alpha_2r_ie^{r_it} + \cdots + \alpha_nr_i^{n-1}e^{r_it} = 0 \text{ for } i = 1, 2, \ldots, n.$$

Or equivalently,

$$e^{r_it}(\alpha_1 + \alpha_2r_i + \cdots + \alpha_nr_i^{n-1}) = 0 \Longleftrightarrow \alpha_1 + \alpha_2r_i + \cdots + \alpha_nr_i^{n-1} = 0 \text{ for } i = 1, 2, \ldots, n. \tag{8.17}$$

Assuming $\boldsymbol{B}$ is singular, there exists nonzero coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$. We set

$$p(r) = \alpha_1 + \alpha_2 r + \cdots + \alpha_n r^{n-1}$$

to be the polynomial of degree $n-1$. On the one hand, we know that a polynomial with degree $n-1$ has $n-1$ roots. On the other hand, the polynomial $p(r)$ has n real and distinct roots $r_1, r_2, \ldots, r_n$, which is a contradiction. Hence $\boldsymbol{B}$ is **nonsingular**, and the Wronskian $\det(\boldsymbol{B})$ is nonzero . ■

We use an example to show how to solve such kind of ODE:

■ **Example 8.3** Find the general solution to

$$y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0 \tag{8.18}$$

Solution. The characteristic equation for Eq.(8.18) is:

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = 0. \tag{8.19}$$

How to solve this equation? If $r = \frac{p}{q}$ is a root for general characteristic equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

where p and q are **coprime**, then p must be a factor of a_n and q must be a factor of a_0. Hence for Eq.(8.19), the factors of a_0 are ± 1, the factors of a_n are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 12$. By testing these possible roots, we find that -2 and 3 are actual roots. Hence we could factorize Eq.(8.19) as:

$$(r-3)(r+2)(r^2 - 6r + 6) = 0$$

Hence $r_1 = -2, r_2 = 3, r_3 = 3 - \sqrt{3}, r_4 = 3 + \sqrt{3}$. The general solution is given by:

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{(3-\sqrt{3})t} + c_4 e^{(3+\sqrt{3})t}.$$

■

■

Some roots of Eq.(8.15) are complex

If some roots of Eq.(8.15) are **complex**, they must occure in pairs, i.e. $\lambda \pm i\mu$. In this case, we could replace the complex-valud solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions:

$$e^{\lambda t} \cos \mu t, \qquad e^{\lambda t} \sin \mu t.$$

■ **Example 8.4** Find the general solution to

$$y^{(4)} - y = 0. \tag{8.20}$$

Solution. The characteristic equation for Eq.(8.20) is:

$$r^4 - 1 = 0. \tag{8.21}$$

We derive that $r = 1, -1, \pm i$. And we take the real and imaginary part of the solution e^{it} to form real-valued solution:

$$e^{it} = \cos t + i \sin t \implies \Re(e^{it}) = \cos t, \quad \Im(e^{it}) = \sin t.$$

Hence $\{e^t, e^{-t}, \cos t, \sin t\}$ forms fundamental set of solutions. The general solution is given by:

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

■

Some roots of Eq.(8.15) are repeated

If one root r_1 of Eq.(8.15) is repeated with multipicity s, then the corresponding solutions to ODE(8.14) is:

$$e^{r_1 t}, te^{r_1 t}, t^2 e^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}.$$

Moreover, if r_1 is complex, i.e. $r_1 = \lambda + i\mu$, then the corresponding conjugate of r_1, $\lambda - i\mu$ is also the root of Eq.(8.15) with multipicity s. In this case, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}, \ldots, t^{s-1}e^{(\lambda-i\mu)t}$ by the real valued solutions by taking the real and imaginary part of $\{e^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}\}$ (or $\{e^{(\lambda-i\mu)t}, \ldots, t^{s-1}e^{(\lambda-i\mu)t}\}$):

$$\begin{aligned} \text{Real part: } & e^{\lambda t}\cos\mu t, te^{\lambda t}\cos\mu t, t^2 e^{\lambda t}\cos\mu t, \ldots, t^{s-1}e^{\lambda t}\cos\mu t \\ \text{Imaginary part: } & e^{\lambda t}\sin\mu t, te^{\lambda t}\sin\mu t, t^2 e^{\lambda t}\sin\mu t, \ldots, t^{s-1}e^{\lambda t}\sin\mu t \end{aligned}$$

■ **Example 8.5** Find the general solution of

$$y^{(4)} + 2y'' + y = 0 \tag{8.22}$$

Solution. The characteristic equation for Eq.(8.22) is:

$$r^4 + 2r^2 + 1 = (r^2+1)(r^2+1) = 0. \tag{8.23}$$

We derive that $r = i, i, -i, -i$. Hence the fundamental solution is:

$$e^{it}, te^{it}, \qquad e^{-it}, te^{-it}$$

We take the real and imaginary part of $\{e^{it}, te^{it}\}$ or $\{e^{-i}, te^{-it}\}$ to form real-valued solution:

$$\begin{aligned} \text{Real part: } & \cos t, t\cos t \\ \text{Imaginary part: } & \sin t, t\sin t \end{aligned}$$

The general solution is given by:

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

How to solve some special characteristic equations ?
Some characteristc equations are difficult to solve. Here we introduce one kind of such equation:

$$r^n + 1 = 0. \tag{8.24}$$

How to solve it? We find the roots of Eq.(8.24) must satisfy

$$r = (-1)^{1/n}.$$

A useful way to compute $(-1)^{1/n}$ is to use Euler's formula $e^{i\pi} + 1 = 0$. It follows that

$$-1 = e^{i\pi} = \cos\pi + i\sin\pi = \cos(\pi + 2m\pi) + i\sin(\pi + 2m\pi = \exp[i(\pi + 2m\pi)]$$

where m is arbitrary integer. Thus

$$(-1)^{1/n} = \exp\left[i\frac{(\pi + 2m\pi)}{n}\right] = \cos\left(\frac{(\pi + 2m\pi)}{n}\right) + i\sin\left(\frac{(\pi + 2m\pi)}{n}\right)$$

The n roots of Eq.(8.24) are obtained by setting $m = 0, 1, 2, \ldots, n-1$:

$$r = \cos\left(\frac{(\pi + 2m\pi)}{n}\right) + i\sin\left(\frac{(\pi + 2m\pi)}{n}\right) \text{ for } m = 0, 1, 2, \ldots n-1.$$

8.2 The method of undetermined coefficients

Here we want to find s particular solution of the **nonhomogeneous** ODE with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{8.25}$$

Before begin to solve it, we should make sure that our $g(t)$ only belong to the class of functions discussed below:

$$\underline{e^{\alpha t}\text{(exponential)}} \qquad \underline{\sin\alpha t \text{ or } \cos\alpha t\text{(triangle)}} \qquad \underline{P_n(t)\text{(polynomial)}}$$

or the **sums** or the **products** of such functions above. Otherwise we **cannot** solve this ODE using this method, we have to use *variation of parameters* discussed in the next lecture!
We want to find the **particular solution** $Y(t)$ to Eq.(8.25) by guessing the forms of $Y(t)$. We can check the table below as the reference:

Note: Here α ***is a "root"*** **means** α ***is a root of the characteristic equation***
$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$

$g(t)$	$Y(t)$	The value for s
$ke^{\alpha t}$	$At^s e^{\alpha t}$	$s = \begin{cases} 0, \text{if } \alpha \text{ is not a root.} \\ m, \text{if } \alpha = r_1, \ r_1 \text{is a root with multiplicity } m \end{cases}$
$P_n(t)e^{\alpha t}$	$Q_n(t)t^s e^{\alpha t}$	**Same as above**
$\begin{cases} P_n e^{\alpha t}\sin\beta t \\ P_n e^{\alpha t}\cos\beta t \end{cases}$	$\begin{cases} [Q_n(t)\cos\beta t \\ +R_n(t)\sin\beta t]t^s e^{\alpha t} \end{cases}$	$s = \begin{cases} 0, \text{if } \alpha + i\beta \text{ is not a root.} \\ m, \text{if } \alpha + i\beta = r_1, \ r_1 \text{ is a root with mutlipliticy} m. \end{cases}$

After guessing the forms of $Y(t)$, we derive $Y', Y'', \ldots, Y^{(n)}$, then we plug them into our Eq.(8.25) to determine the coefficients. Thus we obtain our particular solution $Y(t)$.
The general solution to Eq.(8.25) is obtained as:

$$y = y_c(t) + Y(t)$$

where $y_c(t)$ is the solution to the homogeneous part, $Y(t)$ is the particular solution.
We use two examples to show how to implement this method:

■ **Example 8.6** Find the general solution to the ODE

$$y''' - 3y'' + 4y' - 2y = t^2 e^{2t} \tag{8.26}$$

Proof.

- For its homogeneous part, the characteristic equation is given by:

$$r^3 - 3r^2 + 4r - 2 = 0. \implies (r-1)(r^2 - 2r + 2) = 0.$$

Thus $r_1 = 1, r_2 = 1+i, r_3 = 1-i$. We take the real and imaginary part of the solution $e^{(1+i)t}$:

$$\Re(e^{(1+i)t}) = \Re(e^t(\cos t + i\sin t)) = e^t \cos t$$

$$\Im(e^{(1+i)t}) = \Im(e^t(\cos t + i\sin t)) = e^t \sin t$$

Hence the solution to the homogeneous part is:

$$y_c = c_1 e^t + c_2 e^t \cos t + c_3 e^t \sin t$$

- Then we want to find the particular solution. Checking the table above, we guess the form of $Y(t)$ to be:

$$Y(t) = (At^2 + Bt + C)e^{2t}$$

It follows that

$$\begin{aligned} Y' &= (2At + B)e^{2t} + 2(At^2 + Bt + C)e^{2t} \\ &= [2At^2 + (2A+2B)t + (B+2C)]e^{2t} \end{aligned}$$

$$\begin{aligned} Y'' &= [4At + (2A+2B)]e^{2t} + 2[2At^2 + (2A+2B)t + (B+2C)]e^{2t} \\ &= [4At^2 + (8A+4B)t + (2A+4B+4C)]e^{2t} \end{aligned}$$

$$\begin{aligned} Y''' &= [8At + (8A+4B)]e^{2t} + 2[4At^2 + (8A+4B)t + (2A+4B+4C)]e^{2t} \\ &= [8At^2 + (24A+8B)t + (12A+12B+8C)] \end{aligned}$$

We plug the above formulas to Eq.(8.26) to obtain:

$$\begin{array}{lll} [8At^2] & -3[4At^2] + 4[2At^2] & -2[At^2] = t^2 e^{2t} \\ [(24A+8B)t] & -3[(8A+4B)t] + 4[(2A+2B)t] & -2[Bt] = 0 \\ [(12A+12B+8C)] & -3[(2A+4B+4C)] + 4[(B+2C)] & -2C = 0. \end{array}$$

It follows that

$$\begin{aligned} 8A - 12A + 8A - 2A &= 1 \\ (24A + 8B) - 3(8A + 4B) + 4(2A + 2B) - 2B &= 0 \\ (12A + 12B + 8C) - 3(2A + 4B + 4C) + 4(B + 2C) - 2C &= 0 \end{aligned} \implies \begin{cases} A = \frac{1}{2} \\ B = -2 \\ C = \frac{5}{2} \end{cases}$$

The particular solution is given by:

$$Y(t) = \left(\frac{1}{2}t^2 - 2t + \frac{5}{2}\right) e^{2t}.$$

The general solution is obtained:

$$y = y_c + Y(t) = c_1 e^{2t} + c_2 e^t \cos t + c_3 e^t \sin t + \left(\frac{1}{2}t^2 - 2t + \frac{5}{2}\right) e^{2t}.$$

In fact, if you are not sure whether your particular solution is correct when doing your homework, you can open your MATLAB and enter the code:

```
syms t
myy = '(A*t^2+B*t+C)*exp(2*t)';
mydy = diff(myy,t);
myddy = diff(myy,t,2);
mydddy = diff(myy,t,3);
simple(subs('D3y-3*D2y+4*Dy-2*y=t^2*exp(2*t)',{'y','Dy','D2y','D3y'},{myy,mydy,myddy,mydddy}));
```

The result is

```
collect(t):

(2*A*exp(2*t))*t^2 + (8*A*exp(2*t) + 2*B*exp(2*t))*t + 6*A*exp(2*t) + 4*B*exp(2*t) + 2*C*exp(2*t) == exp(2*t)*t^2
```

Hence you only need to solve linear system of equation. Based on the result, you enter

```
[A,B,C] = solve('2*A=1','8*A+2*B=0','6*A+4*B+2*C=0','A','B','C')
```

You will get the corrsponding coefficients

$$A = \frac{1}{2}, \quad B = -2, \quad C = \frac{5}{2}$$

which agrees with our result. ■

■ **Example 8.7** Find the general solution to the ODE

$$y''' - 3y'' + 2y' = t + e^t = t \cdot e^{0t} + e^{2t} \tag{8.27}$$

Solution. • For the homogeneous part, the characteristic equation is given by:

$$r^3 - 3r^2 + 2r = r(r-1)(r-2) = 0$$

Hence $r_1 = 0, r_2 = 1, r_3 = 2$. The solution to the homogeneous part is:

$$y_c(t) = c_1 + c_2 e^t + c_3 e^{2t}.$$

• Then we want to find the particular solution. Note that 0 is a root of the characteristic equation. Hence we guess the first term of $Y(t)$ is $(At+B)t$. Since 2 is a root of the characteristic, we guess the second term of $Y(t)$ is Cte^{2t}. Hence

$$Y(t) = (At+B)t + Cte^{2t} = At^2 + Bt + Cte^{2t}.$$

It follows that

$$Y' = 2At + B + C(2t+1)e^{2t}$$

$$Y'' = 2A + C(4t+4)e^{2t}$$

$$Y''' = C(8t+12)e^{2t}$$

We plug the above formulas to Eq.(8.27) to obtain:

$$\begin{aligned} C(8t+12)e^{2t} - 3C(4t+4)e^{2t} + 2C(2t+1)e^{2t} &= e^{2t} \\ 0 - 3\cdot(2A) + 2\cdot(2At+B) &= t \end{aligned}$$

It follows that

$$\begin{aligned} 8C - 12C + 4C &= 0 \\ 12C - 12C + 2C &= 1 \\ 4A &= 1 \\ -6A + 2B &= 0 \end{aligned} \implies \begin{cases} A = \dfrac{1}{4} \\ B = \dfrac{3}{4} \\ C = \dfrac{1}{2} \end{cases}$$

Thus the particular solution is given by:

$$Y(t) = \frac{1}{4}t^2 + \frac{3}{4}t + \frac{1}{2}te^{2t}$$

The general solution is given by:

$$y = y_c + Y = c_1 + c_2 e^t + c_3 e^{2t} + \frac{1}{4}t^2 + \frac{3}{4}t + \frac{1}{2}te^{2t}.$$

■

8.3 Variation of parameters

We could use variation of parameters to determine a particular solution of the inhomogeneous nth order **linear** differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{8.28}$$

■ **Solution 8.1** Here we want to find the particular solution to Eq.(8.28). Suppose we have known a fundamental set of solutions $\{y_1, y_2, \dots, y_n\}$ to the corresponding homogeneous ODE. Then the general solution to the homogeneous equation is:

$$y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

The idea of variation of parameters is that we guess our particular solution to be:

$$Y(t) = C_1(t)y_1 + C_2(t)y_2 + \cdots + C_n(t)y_n \tag{8.29}$$

We take the derivative for $Y(t)$ to obtain:

$$\begin{aligned} Y' &= [C_1'y_1 + C_1y_1'] + [C_2'y_2 + C_2y_2'] + \cdots + [C_n'y_n + C_ny_n'] \\ &= [C_1y_1' + C_2y_2' + \cdots + C_ny_n'] + [C_1'y_1 + C_2'y_2 + \cdots + C_n'y_n] \end{aligned}$$

And we **set** the first condition:

$$C_1'y_1 + C_2'y_2 + \cdots + C_n'y_n = 0. \tag{8.30}$$

It follows that

$$Y' = C_1y_1' + C_2y_2' + \cdots + C_ny_n'$$

And we continue to calculate $Y'', \dots, Y^{(n-1)}$ recursively. After each differentiation, we set the sum of terms invoving $C_1', C_2', \dots, C_n'$ to be zero, i.e.

$$C_1'y_1^{(k-1)} + C_2'y_2^{(k-1)} + \cdots + C_n'y_n^{(k-1)} = 0. \text{ for } k = 1, 2, \dots, n-1. \tag{8.31}$$

As a result, the kth order derivative for $Y(t)$ could be expressed as:

$$Y^{(k)}(t) = C_1y_1^{(k)} + C_2y_2^{(k)} + \cdots + C_ny_n^{(k)} \text{ for } k = 1, 2, \dots, n-1.$$

It follows that

$$Y^{(n)}(t) = [C_1y_1^{(n)} + C_2y_2^{(n)} + \cdots + C_ny_n^{(n)}] + [C_1'y_1^{(n-1)} + C_2'y_2^{(n-1)} + \cdots + C_n'y_n^{(n-1)}]$$

Substituting $y, y', \dots, y^{(n)}$ in Eq.(8.28), we derive:

$$\begin{aligned}
y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)& \\
&= [C_1 y_1^{(n)} + C_2 y_2^{(n)} + \cdots + C_n y_n^{(n)}] + p_1(t)[C_1 y_1^{(n-1)} + C_2 y_2^{(n-1)} + \cdots + C_n y_n^{(n-1)}] \\
&+ p_2(t)[C_1 y_1^{(n-2)} + C_2 y_2^{(n-2)} + \cdots + C_n y_n^{(n-2)}] + \dots \\
&+ p_n(t)[C_1 y_1 + C_2 y_2 + \cdots + C_n y_n] + [C_1' y_1^{(n-1)} + C_2' y_2^{(n-1)} + \cdots + C_n' y_n^{(n-1)}] \\
&= C_1[y_1^{(n)} + p_1(t) + y_1^{(n-1)} + \cdots + p_{n-1}(t)y_1' + p_n(t)y_1] + \cdots \\
&+ C_n[y_n^{(n)} + p_1(t) + y_n^{(n-1)} + \cdots + p_{n-1}(t)y_n' + p_n(t)y_n] \\
&+ [C_1' y_1^{(n-1)} + C_2' y_2^{(n-1)} + \cdots + C_n' y_n^{(n-1)}] \\
&= \sum_{m=1}^{n} C_m[y_m^{(n)} + p_1(t) + y_m^{(n-1)} + \cdots + p_{n-1}(t)y_m' + p_n(t)y_m] \\
&+ [C_1' y_1^{(n-1)} + C_2' y_2^{(n-1)} + \cdots + C_n' y_n^{(n-1)}]
\end{aligned}$$

Since $y_1, y_2, \dots, y_n$ are solutions to the homogeneous part, Hence the term for summation in the equation above is **zero**. Hence

$$\begin{aligned}
y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)& \\
&= C_1' y_1^{(n-1)} + C_2' y_2^{(n-1)} + \cdots + C_n' y_n^{(n-1)}
\end{aligned}$$

Thus we only need to solve the equation

$$g(t) = C_1' y_1^{(n-1)} + C_2' y_2^{(n-1)} + \cdots + C_n' y_n^{(n-1)} \tag{8.32}$$

We write the Eq.(8.31) and Eq.(8.32) together into compact matrix form:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix} \tag{8.33}$$

We apply cramer's rule to get C_i's in Eq.(8.33):

$$C_i = \frac{\boldsymbol{A}_i}{W(t)} \text{ for } i = 1, 2, \dots, n$$

where $\boldsymbol{A}_i$ is the determinant obtained by replacing the ith column of $W(t)$ by the column $(0, 0, 0, \dots, g)$. $W(t)$ is the Wronskian of Eq.(8.28).
Moreover, to simplifty computation, we write $\boldsymbol{A}_i$ as the product of $W_i(t)$ and $g(t)$:

$$\boldsymbol{A}_i = g(t)W_i(t)$$

where $W_i(t)$ is the determinant obtained by replacing the ith column of $W(t)$ by the column $(0,0,0,\ldots,1)$.
Hence we derive:

$$C_i'(t) = \frac{g(t)W_i(t)}{W(t)} \implies C_i(t) = \int \frac{g(s)W_i(s)}{W(s)}\,\mathrm{d}s \text{ for } i = 1,2,\ldots,n$$

Hence our particular solution is given by:

$$Y(t) = \sum_{i=1}^{n} C_i(t)y_i(t).$$

■

The part discussed above could be summarize into one theorem, which can be used directly in exam:

Theorem 8.4 Assume that $p_1, p_2, \ldots, p_n$ and g are c.n.t. on an open interval I, and $\{y_1, y_2, \ldots, y_n\}$ are a fundamental set of solutions to homogeneous Eq.(8.28), then a **particular solution** to Eq.(8.28) is

$$y_0(t) = \sum_{i=1}^{n}\left[\int \frac{g(s)W_i(s)}{W(s)}\,\mathrm{d}s\right] y_i$$

The general solution is given by

$$y(t) = \underbrace{c_1y_1 + c_2y_2 + \cdots + c_ny_n}_{\text{solution to homogeneous ODE}} + \underbrace{y_0(t)}_{\text{particular solution}}$$

for arbitrary constants $c_1, c_2, \ldots, c_n$.

Here we show two examples on how to sovle inhomogeneous high order linear ODE:

■ **Example 8.8** Find the general solution to equation:

$$y''' - 3y'' + 4y' - 2y = \frac{e^t}{\cos t}, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{8.34}$$

(Hint: you may use the formula $\int \frac{1}{\cos t}\,\mathrm{d}t = \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right| + C$.)
It is easy to verify that the general solution to the corresponding homogeneous ODE is:

$$y_c = c_1e^t + c_2e^t\cos t + c_3e^t\sin t.$$

The formula for the particular solution is:

$$y_0(t) = \sum_{i=1}^{3}\left[\int \frac{g(s)W_i(s)}{W(s)}\,\mathrm{d}s\right] y_i$$

By the Abel's theorem (Theorem(8.2)), we compute the Wronskian $W(t)$:

$$W(t) = \exp\left[-\int(-3)\,\mathrm{d}t\right] = e^{3t}.$$

Or you may calculate the Wronskian directly:

$$W(t) = \begin{vmatrix} e^t & e^t\cos t & e^t\sin t \\ e^t & e^t(\cos t - \sin t) & e^t(\sin t + \cos t) \\ e^t & e^t(-2\sin t) & e^t(2\cos t) \end{vmatrix} = e^{3t} \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & \cos t - \sin t & \sin t + \cos t \\ 1 & -2\sin t & 2\cos t \end{vmatrix} = e^{3t}$$

And we derive:

$$W_1(t) = \begin{vmatrix} 0 & e^t\cos t & e^t\sin t \\ 0 & e^t(\cos t - \sin t) & e^t(\sin t + \cos t) \\ 1 & e^t(-2\sin t) & e^t(2\cos t) \end{vmatrix} = e^{2t}.$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^t\sin t \\ e^t & 0 & e^t(\sin t + \cos t) \\ e^t & 1 & e^t(2\cos t) \end{vmatrix} = -e^{2t}\cos t$$

$$W_3(t) = \begin{vmatrix} e^t & e^t\cos t & 0 \\ e^t & e^t(\cos t - \sin t) & 0 \\ e^t & e^t(-2\sin t) & 1 \end{vmatrix} = -e^{2t}\sin t$$

It follows that

$$\begin{aligned} Y(t) &= e^t \int \frac{\frac{e^t}{\cos t}e^{2t}}{e^{3t}}\,\mathrm{d}t + e^t\cos t \int \frac{\frac{e^t}{\cos t}\cdot(-e^{2t}\cos t)}{e^{3t}}\,\mathrm{d}t + e^t\sin t \int \frac{\frac{e^t}{\cos t}\cdot(-e^{2t}\sin t)}{e^{3t}}\,\mathrm{d}t \\ &= e^t \int \frac{1}{\cos t}\,\mathrm{d}t + e^t\cos t \int (-1)\,\mathrm{d}t + e^t\sin t \int (-\tan t)\,\mathrm{d}t \\ &= e^t\cdot\frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right| - te^t\cos t + e^t\sin t\ln|\cos t| \end{aligned}$$

Since $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain our particular solution:

$$Y(t) = e^t\cdot\frac{1}{2}\ln\left(\frac{1+\sin t}{1-\sin t}\right) - te^t\cos t + e^t\sin t\ln(\cos t)$$

Hence our general solution is given by:

$$\begin{aligned} y &= y_c(t) + Y(t) \\ &= c_1e^t + c_2e^t\cos t + c_3e^t\sin t + e^t\cdot\frac{1}{2}\ln\left(\frac{1+\sin t}{1-\sin t}\right) - te^t\cos t + e^t\sin t\ln(\cos t) \end{aligned}$$

■

■ **Example 8.9** Find the solution to the IVP:

$$y''' - y'' + y' - y = g(t), \qquad y(0) = 0, y'(0), y''(0) = 0. \tag{8.35}$$

It is easy to verify that the geneous solution to the corresponding homogeneous ODE is

$$y_c = c_1e^t + c_2\cos t + c_3\sin t.$$

The formula for the particular solution is:

$$y_0(t) = \sum_{i=1}^{3} \left[\int \frac{g(s)W_i(s)}{W(s)} \, \mathrm{d}s \right] y_i$$

By Abel's theorem, the Wronskian is obtained:

$$W(t) = \exp\left[- \int (-1) \, \mathrm{d}t \right] = e^t$$

Or you may calculate it directly:

$$W(t) = \begin{vmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{vmatrix} = \begin{vmatrix} e^t & \cos t & \sin t \\ 0 & -\sin t - \cos t & \cos t - \sin t \\ 0 & -2\cos t & -2\sin t \end{vmatrix} = 2e^t$$

And we derive:

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & \sin t \\ e^t & 0 & \cos t \\ e^t & 1 & -\sin t \end{vmatrix} = e^t(\sin t - \cos t)$$

$$W_3(t) = \begin{vmatrix} e^t & \cos t & 0 \\ e^t & -\sin t & 0 \\ e^t & -\cos t & 1 \end{vmatrix} = e^t(-\sin t - \cos t)$$

It follows that

$$\begin{aligned} Y(t) &= e^t \int \frac{g(t)\cdot 1}{2e^t} \, \mathrm{d}t + \cos t \int \frac{g(t)\cdot e^t(\sin t - \cos t)}{2e^t} \, \mathrm{d}t + \sin t \int \frac{g(t)\cdot e^t(-\sin t - \cos t)}{2e^t} \, \mathrm{d}t \\ &= \frac{1}{2}e^t \int g(t)e^{-t} \, \mathrm{d}t + \frac{1}{2}\cos t \int g(t)(\sin t - \cos t) \, \mathrm{d}t - \frac{1}{2}\sin t \int g(t)(\sin t + \cos t) \, \mathrm{d}t \end{aligned}$$

Hence the general solution to Eq.(8.35) is given by:

$$\begin{aligned} y &= y_0(t) + Y(t) \\ &= c_1 e^t + c_2 \cos t + c_3 \sin t + \frac{1}{2}e^t \int g(t)e^{-t} \, \mathrm{d}t \\ &+ \frac{1}{2}\cos t \int g(t)(\sin t - \cos t) \, \mathrm{d}t - \frac{1}{2}\sin t \int g(t)(\sin t + \cos t) \, \mathrm{d}t \end{aligned}$$

And we find that

$$y(0) = c_1 + c_2 + \frac{1}{2}\left[\int g(t)e^{-t} \, \mathrm{d}t \right]_{t=0} + \frac{1}{2}\left[\int g(t)(\sin t - \cos t) \, \mathrm{d}t \right]_{t=0} = 0.$$

Warning: Remember that you should consider the indefinite integration as the function with respect to t and think about the derivative of this indefinite integration.

$$
\begin{aligned}
y'(0) &= c_1 + c_3 + \frac{1}{2}\left[\int g(t)e^{-t}\,\mathrm{d}t\right]_{t=0} + \frac{1}{2}\left[g(0)e^{-0}\,\mathrm{d}t\right] \\
&\quad + \frac{1}{2}\left[g(0)(0-1)\right] - \frac{1}{2}\left[\int g(t)(\sin t + \cos t)\,\mathrm{d}t\right]_{t=0} \\
&= c_1 + c_3 + \frac{1}{2}\left[\int g(t)e^{-t}\,\mathrm{d}t\right]_{t=0} - \frac{1}{2}\left[\int g(t)(\sin t + \cos t)\,\mathrm{d}t\right]_{t=0} = 0.
\end{aligned}
$$

$$
y''(0) = c_1 - c_2 + \frac{1}{2}\left[\int g(t)e^{-t}\,\mathrm{d}t\right]_{t=0} - \frac{1}{2}\left[\int g(t)(\sin t - \cos t)\,\mathrm{d}t\right]_{t=0} = 0.
$$

It follows that

$$
\begin{cases}
c_1 = -\dfrac{1}{2}\left[\displaystyle\int g(t)e^{-t}\,\mathrm{d}t\right]_{t=0} \\
c_2 = -\dfrac{1}{2}\left[\displaystyle\int g(t)(\sin t - \cos t)\,\mathrm{d}t\right]_{t=0} \\
c_3 = \dfrac{1}{2}\left[\displaystyle\int g(t)(\sin t + \cos t)\,\mathrm{d}t\right]_{t=0}
\end{cases}
$$

Thus the solution to the IVP is given by:

$$
\begin{aligned}
y &= -\frac{1}{2}e^t\left[\int g(t)e^{-t}\,\mathrm{d}t\right]_{t=0} - \frac{1}{2}\cos t\left[\int g(t)(\sin t - \cos t)\,\mathrm{d}t\right]_{t=0} + \frac{1}{2}\sin t\left[\int g(t)(\sin t + \cos t)\,\mathrm{d}t\right]_{t=0} \\
&\quad + \frac{1}{2}e^t\int g(t)e^{-t}\,\mathrm{d}t + \frac{1}{2}\cos t\int g(t)(\sin t - \cos t)\,\mathrm{d}t - \frac{1}{2}\sin t\int g(t)(\sin t + \cos t)\,\mathrm{d}t \\
&= \frac{1}{2}e^t\left\{\int g(t)e^{-t}\,\mathrm{d}t - \left[\int g(t)e^{-t}\,\mathrm{d}t\right]_{t=0}\right\} \\
&\quad + \frac{1}{2}\cos t\left\{\int g(t)(\sin t - \cos t)\,\mathrm{d}t - \left[\int g(t)(\sin t - \cos t)\,\mathrm{d}t\right]_{t=0}\right\} \\
&\quad - \frac{1}{2}\sin t\left\{\int g(t)(\sin t + \cos t)\,\mathrm{d}t - \left[\int g(t)(\sin t + \cos t)\,\mathrm{d}t\right]_{t=0}\right\} \\
&= \frac{1}{2}e^t\int_0^t g(s)e^{-s}\,\mathrm{d}s + \frac{1}{2}\cos t\int_0^t g(s)(\sin s - \cos s)\,\mathrm{d}s - \frac{1}{2}\sin t\int_0^t g(s)(\sin s + \cos s)\,\mathrm{d}s \\
&= \frac{1}{2}\int_0^t g(s)e^{t-s} - \sin(t-s) - \cos(t-s)\,\mathrm{d}s
\end{aligned}
$$

■

9 — Week9

9.1 Introduction to 1st order system ODE

The nth order linear ODE could be transformed into 1st order linear system ODE. For example, given the ODE

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{9.1}$$

we want to reduce it into a system, setting $x_1 = y, x_2 = y', \ldots, x_n = y^{n-1}$, we could transform this high order ODE as:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \quad\vdots \\ \quad\vdots \\ x_{n-1}' = x_n \\ x_n' = -p_1(t)x_n - p_2(t)x_{n-1} - \cdots - p_n(t)x_1 + g(t) \end{cases}$$

Or equivalently, we could write this ststem as compact matrix form:

$$\boldsymbol{x}' = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{g}(t) \tag{9.2}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$$

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & -p_{n-3}(t) & -p_{n-4}(t) & \cdots & -p_1(t) \end{bmatrix}$$

$$\boldsymbol{g}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & g(t) \end{bmatrix}^{\mathrm{T}}.$$

One reason why we study system of first order equations is that equations of high order can always be transformed into systems. So if you face lots of calculation for a high order ODE, transforming it into system and using techniques related to systems to solve it is a goog idea.

Definition 9.1 The general form of first order ODE could be expressed as:

$$\begin{cases} x_1' = F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' = F_2(t, x_1, x_2, \ldots, x_n) \\ \cdots \\ x_n' = F_n(t, x_1, x_2, \ldots, x_n) \end{cases} \tag{9.3}$$

where $x_i: \mathbb{C} \mapsto \mathbb{C}$ is a function of t, $F_i: \mathbb{C}^{n+1} \mapsto \mathbb{C}$ for $i = 1, 2, \ldots, n$. $t \in \mathbb{C}$.
Also, we could write the system into matrix form:

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}). \tag{9.4}$$

where $\boldsymbol{x}: \mathbb{C} \mapsto \mathbb{C}^{n\times 1}$ is a function of t, $f: \mathbb{C}^{n+1} \mapsto \mathbb{C}^n$ is a function of t and $\boldsymbol{x}$. ■

Consider the IVP:

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}), \qquad \boldsymbol{f}(t_0) = \boldsymbol{c}_0 \tag{9.5}$$

When does such problem exist a unqiue solution? We consider this problem under the assumption below:
Assumption: On a **rectangle** region $R = \{(t, \boldsymbol{x}): \ |t - t_0| \le a, \ |\boldsymbol{x} - \boldsymbol{x}_0| \le b\}$, where a and b are two positive numbers, we have:

1. $t \in \mathbb{R}$.
2. All entries of $\boldsymbol{x}$ and $\boldsymbol{f}(t, \boldsymbol{x})$ are real-valued.
3. All entries of $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$ are **continuous**. i.e.

$$\frac{\partial F_i}{\partial x_j} \text{ are continuous for } i, j \in \{1, 2, \ldots, n\}.$$

Then we obtain the **existence and uniqueness theorem**:

Theorem 9.1 — Existence and Uniqueness theorem.
Under the assumption above, there exists a unqiue solution $\boldsymbol{x}^0 = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$ of Eq.(9.5) on the interval $|t - t_0| < a$.

The proof for this theorem is skipped.
It is easy to find solutions to a specific kind of system of ODE. That is the first order **lienar** ODE:

Definition 9.2 The general form of first order **linear** ODE could be expressed as:

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\ \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases} \tag{9.6}$$

Also, we could write it into compact matrix form:

$$\boldsymbol{x}' = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{g}(t) \tag{9.7}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$$

$$\boldsymbol{A} = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix}$$

$$\boldsymbol{g}(t) = \begin{bmatrix} g_1(t) & g_2(t) & \cdots & g_n(t) \end{bmatrix}^{\mathrm{T}}$$

If $g_1(t) = g_2(t) = \cdots = g_n(t) = 0$, the system(9.6) is called **homogeneous** system, otherwise it is **inhomogeneous**.
In the following lectures we will talk about how to find solutions to such kind of system of ODEs.
But now let's talk about when does this system exist unique solution first.
Consider the IVP:

$$\boldsymbol{x}' = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{g}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{9.8}$$

Assumption: On a open interval $I = \{t : \alpha < t < \beta\}$, where α and β are two real nu,bers, we have:

All entries of $\boldsymbol{A}(t)$ and all entries of $\boldsymbol{g}(t)$ are continuous

Then we obtain the **existence and uniqueness theorem**:

Theorem 9.2 — Existence and Uniqueness theorem for linear system.
Under the assumption above, there exists a unqiue solution $\boldsymbol{x}^0 = \begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{bmatrix}$ of the Eq.(9.8) on the interval $I : \alpha < t < \beta$.

9.2 Review of Matrices

9.2.1 *Linearly dependent/independent vectors*

Consider k vectors

$$\boldsymbol{a}^1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad , \cdots, \quad , \boldsymbol{a}^k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}$$

A collection of vectors $\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^k$ is said to be **linearly dependent** if $\exists$ constants $x_1, \ldots, x_k \in \mathbb{R}^n$ that are not all zero s.t.

$$x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \cdots + x_k \boldsymbol{a}_k = \boldsymbol{0}.$$

Otherwise they are **linearly independent**.
Moreover, $\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^k$ is **linearly dependent** if and only if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has **non-zero** solution, where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}^1 & \boldsymbol{a}^2 & \ldots & \boldsymbol{a}^k \end{bmatrix},$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \end{pmatrix}.$$

9.2.2 *Inverse, nonsingular / invertible*

Square matrix $\boldsymbol{A}$ is called **nonsingular** or **invertible** if $\exists$ $\boldsymbol{B}$ s.t.

$$\boldsymbol{AB} = \boldsymbol{I} \text{ and } \boldsymbol{BA} = \boldsymbol{I}$$

where $\boldsymbol{I}$ is **identity** matrix.
If $\boldsymbol{A}$ is nonsingular, its inverse is denoted by $\boldsymbol{A}^{-1}$. Then we have

$$\boldsymbol{Ax} = \boldsymbol{0} \implies \boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{0} \implies \boldsymbol{x} = \boldsymbol{0}$$

Thus column vectors of $\boldsymbol{A}$ are **linearly independent**.

Theorem 9.3 These statements are equivalent:

1. $\boldsymbol{A}$ is nonsingular
2. $\boldsymbol{Ax} = \boldsymbol{0}$ has zero solution only
3. $\boldsymbol{Ax} = \boldsymbol{b}$ has unique solution
4. column vectors of $\boldsymbol{A}$ are **linearly independent**.
5. $\det(\boldsymbol{A}) \neq 0$.

Theorem 9.4
These statements are equivalent:

1. $\boldsymbol{A}$ is singular
2. $\boldsymbol{Ax} = \boldsymbol{0}$ has nonzero solution
3. column vectors of $\boldsymbol{A}$ are **linear dependent**
4. $\det(\boldsymbol{A}) = 0$.

9.2.3 Determinant

We list some nice properties about **determinant**:

- $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ for any $\boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{m\times m}$.
- $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$.
- $\det(\alpha\boldsymbol{A}) = \alpha^m \det(\boldsymbol{A})$ for any $\alpha \in \mathbb{R}, \boldsymbol{A} \in \mathbb{R}^{m\times m}$.
- $\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$ for any **nonsingular** $\boldsymbol{A}$.
- $\det(\boldsymbol{B}^{-1}\boldsymbol{AB}) = \det(\boldsymbol{A})$ for any nonsingular $\boldsymbol{B}$.
- If $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is triangular, either upper or lower,

$$\det(\boldsymbol{A}) = \prod_{i=1}^{m} a_{ii}$$

Proof. Apply cofactor expansion inductively. ∎

- If $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ takes a block upper triangular form:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix}$$

where $\boldsymbol{B}$ and $\boldsymbol{D}$ are square, then

$$\det(\boldsymbol{A}) = \det(\boldsymbol{B})\det(\boldsymbol{D}).$$

9.2.4 Gaussian Elimination

The inverse of a matrix could be derived by elementary row operations:

$$[\boldsymbol{A}|\boldsymbol{I}] \Longrightarrow \text{elementary row operations}[\boldsymbol{I}|\boldsymbol{A}^{-1}]$$

Let me show you an example:

■ **Example 9.1** Find the inverse of

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

Solution.

$$[\boldsymbol{A}|\boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ -2 & -2 & 3 & 0 & 1 & 0 \\ 2 & 4 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Add } (-2)\times\text{Row 1 to Row 3}]{\text{Add } 2\times\text{Row 1 to Row 2}} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 2 & -3 & 2 & 1 & 0 \\ 0 & 0 & 12 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Row } 3\times\frac{1}{12}]{\text{Row } 2\times\frac{1}{2}} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & 0 & \frac{1}{12} \end{array}\right] \xrightarrow[\text{Add } \frac{3}{2}\times\text{Row 3 to Row 2}]{\text{Add } 3\times\text{Row 3 to Row 1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} \\ 0 & 0 & 1 & -\frac{1}{6} & 0 & \frac{1}{12} \end{array}\right]$$

$$\xrightarrow{\text{Add } (-2)\times\text{Row 2 to Row 1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} \\ 0 & 0 & 1 & -\frac{1}{6} & 0 & \frac{1}{12} \end{array}\right]$$

∎

9.2.5 Eigenvalues & Eigenvectors

Definition 9.3 We want to study a **Key Problem**:
Given a $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$), we want to find a vector $\boldsymbol{v} \in \mathbb{C}^n$ with $\boldsymbol{v} \neq \boldsymbol{0}$ such that

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \text{ for some } \lambda \in \mathbb{C} \tag{9.9}$$

- (9.9) is called an **eigenvalue problem** or **eigen-equation**.
- Let $(\boldsymbol{v}, \lambda)$ be a solution to Eq.(9.9), we say
  - $(\boldsymbol{v}, \lambda)$ is an **eigen-pair** of $\boldsymbol{A}$.
  - λ is an **eigenvalue** of $\boldsymbol{A}$; $\boldsymbol{v}$ is an **eigenvector** of $\boldsymbol{A}$ associated with λ
- If $(\boldsymbol{v}, \lambda)$ is an eigen-pair of $\boldsymbol{A}$, $(\alpha\boldsymbol{v}, \lambda)$ is also an eigen-pair of $\boldsymbol{A}$ associatd with λ.

But how to find eigenvalues and eigenvectors?

- From the eigenvalue problem we see that

$$\begin{aligned} \boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} \text{ for some } \boldsymbol{v} \neq \boldsymbol{0} &\iff (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \text{ for some } \boldsymbol{v} \neq \boldsymbol{0}. \\ &\iff \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0 \end{aligned}$$

  We let $p(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I})$, which is called the **characteristic polynomial of $\boldsymbol{A}$**. Solving for $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$, which is called the **characteristic equation**, we could derive n eigenvalues of $\boldsymbol{A}$.
- For n eigenvalues of $\boldsymbol{A}$, $\lambda_1, \lambda_2, \dots, \lambda_n$, we can solve for

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i \text{ for } i = 1, 2, \dots, n$$

  to derive eigenvectors associated with n eigenvalues.

■ **Example 9.2** Let's try to derive the eigenvalue and eigenvector of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation is given by:

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) = 0.$$

Thus $\lambda_1 = 1, \lambda_2 = 2$.
When $\lambda = 1$, we have to solve

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0} \implies \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{v}_1 = \boldsymbol{0} \implies \boldsymbol{v}_1 = c\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

When $\lambda = 2$, we have to solve

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0} \implies \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\boldsymbol{v}_2 = \boldsymbol{0} \implies \boldsymbol{v}_2 = c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence $(\lambda_1, \boldsymbol{v}_1)$ and $(\lambda_2, \boldsymbol{v}_2)$ are two eigen-pairs of $\boldsymbol{A}$. ■

■ **Example 9.3** Fact: Eigenvalues and eigenvectors of $\boldsymbol{A}$ could be complex even if $\boldsymbol{A}$ is real. For example, consider

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We derive $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^2 + 1 = 0$ Hence $\lambda_1 = i, \lambda_2 = -i$.
When $\lambda = i$, we find

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0} \implies \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \boldsymbol{v}_1 = \boldsymbol{0} \implies \boldsymbol{v}_1 = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

When $\lambda = -i$, we find

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0} \implies \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \boldsymbol{v}_2 = \boldsymbol{0} \implies \boldsymbol{v}_2 = c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Hence $(i, \boldsymbol{v}_1)$ and $(-i, \boldsymbol{v}_2)$ are two eigen-pairs of $\boldsymbol{A}$. ■

Definition 9.4 — Multiplicity.
Suppose matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has **distinct** eigenvalues λ_i for $i = 1, 2, \ldots, k$.

- The **algebraic multiplicity** of an eigenvalue $\lambda_i, i \in \{1, 2, \ldots, k\}$ is defined as the *number of times that λ_i appears as a root of the* $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$. We denote the *algebraic multiplicity of λ_i* as m_i. In other words, we denote m_i as the number of repeated eigenvalues of λ_i.
- The **geometric multiplicity** of an eigenvalue $\lambda_i, i \in \{1, 2, \ldots, k\}$ is defined as the *maximal number of* ***linearly independent*** *eigenvectors associated with λ_i*. And we denote the *geometric multiplicity of λ_i* as q_i. Note that $q_i = \dim(N(\boldsymbol{A} - \lambda_i \boldsymbol{I}))$.

■

Proposition 9.1 We have $m_i \geq q_i$ for $i = 1, 2, \ldots, k$.

The implication is that **The number of repeated eigenvalues of $\lambda_i \geq$ The number of linearly independent eigenvectors associated with λ_i.**
And $m_i > q_i$ is possible, let's raise an example:

■ **Example 9.4**

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can verify that the roots of $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$ are $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus we have $k = 1, m_1 = 3$.

However, we can also verify that

$$N(\lambda-\lambda_1\boldsymbol{I})=N(\boldsymbol{A})=\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

And consequently, $q_1=\dim(N(\boldsymbol{A}-\lambda_1\boldsymbol{I}))=2$. Thus $m_1>q_1$.

Proof for proposition. For convenience, we let $\lambda_0\in\{\lambda_1,\dots,\lambda_k\}$ be any eigenvalue of $\boldsymbol{A}$, and we denote $q=\dim(N(\boldsymbol{A}-\lambda_0\boldsymbol{I}))$. We only need to show that $\det(\boldsymbol{A}-\lambda\boldsymbol{I})$ has at least q repeated roots for $\lambda=\lambda_0$.

Firstly, let's focus on **real** eigenvalues and **real** eigenvectors:

- From concepts for subspace, we can find a collection of **orthonormal** vectors $\boldsymbol{v}_1,\dots,\boldsymbol{v}_q\in N(\boldsymbol{A}-\lambda_0\boldsymbol{I})$ and a collection of vectors $\boldsymbol{v}_{q+1},\dots,\boldsymbol{v}_n\in\mathbb{R}^n$ such that

$$\boldsymbol{V}=\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix}\text{ is orthogonal.}$$

Let $\boldsymbol{V}_1=\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_q\end{bmatrix}$, $\boldsymbol{V}_2=\begin{bmatrix}\boldsymbol{v}_{q+1} & \boldsymbol{v}_{q+2} & \cdots & \boldsymbol{v}_n\end{bmatrix}$ and note $\boldsymbol{V}=\begin{bmatrix}\boldsymbol{V}_1 & \boldsymbol{V}_2\end{bmatrix}$. Thus we have

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}=\begin{bmatrix}\boldsymbol{V}_1^{\mathrm{T}}\\\boldsymbol{V}_2^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{A}\boldsymbol{V}_1 & \boldsymbol{A}\boldsymbol{V}_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_1 & \boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2\\\boldsymbol{V}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_1 & \boldsymbol{V}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2\end{bmatrix}$$

Since $\boldsymbol{A}\boldsymbol{v}_i=\lambda_0\boldsymbol{v}_i$ for $i=1,2,\dots,q$, we get $\boldsymbol{A}\boldsymbol{V}_1=\lambda_0\boldsymbol{V}_1$. By also noting that $\boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{V}_1=\boldsymbol{I}$ and $\boldsymbol{V}_2^{\mathrm{T}}\boldsymbol{V}_1=\boldsymbol{0}$, we can simplify the above matrix equation into:

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}=\begin{bmatrix}\lambda_0\boldsymbol{I} & \boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2\\\boldsymbol{0} & \boldsymbol{V}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2\end{bmatrix}$$

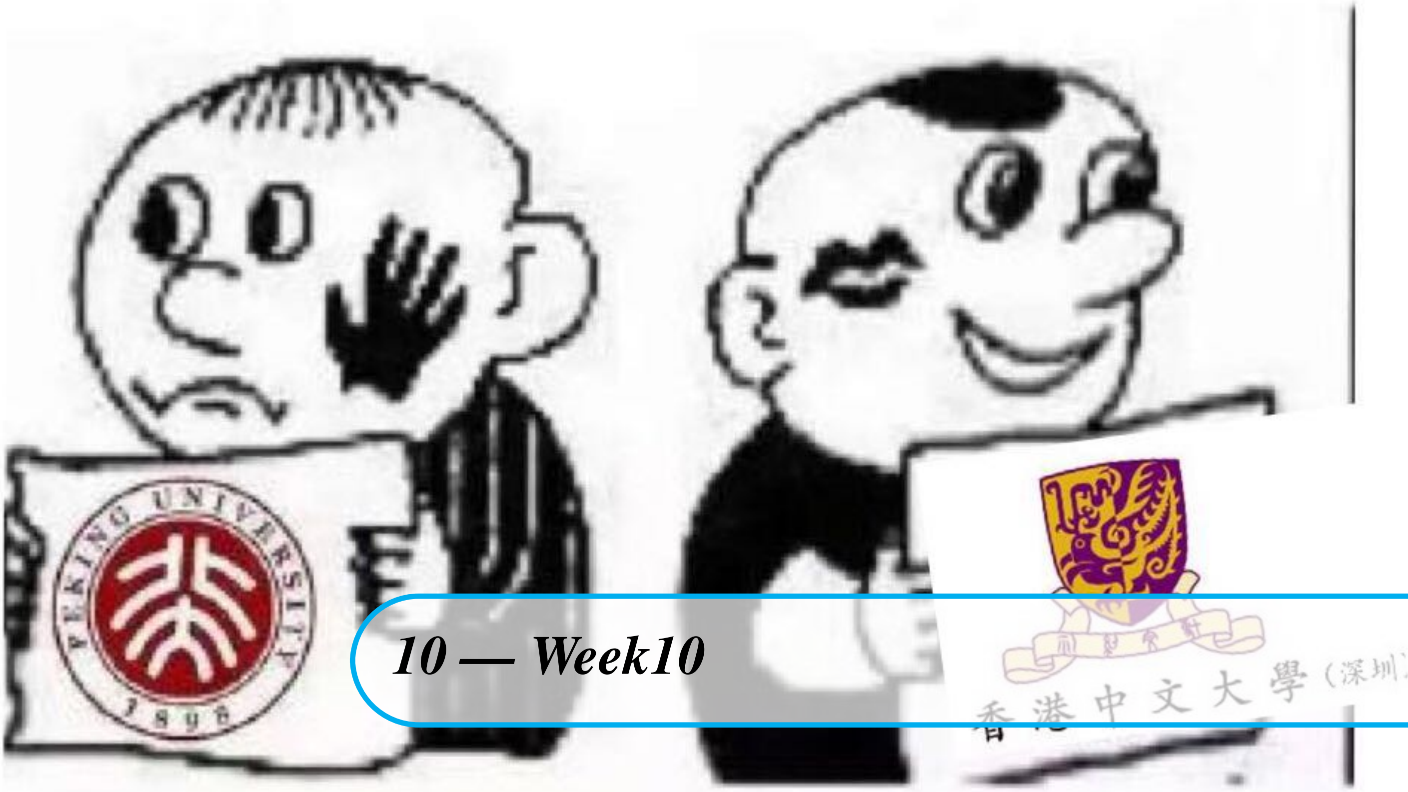
It follows that

$$\begin{aligned}\det(\boldsymbol{A}-\lambda\boldsymbol{I})&=\det(\boldsymbol{V}^{\mathrm{T}}(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{V})=\det(\boldsymbol{V}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}-\lambda\boldsymbol{I})\\&=\det\begin{pmatrix}(\lambda_0-\lambda)\boldsymbol{I} & \boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2\\\boldsymbol{0} & \boldsymbol{V}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2-\lambda\boldsymbol{I}\end{pmatrix}\\&=(\lambda_0-\lambda)^q\det(\boldsymbol{V}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2-\lambda\boldsymbol{I})\end{aligned}$$

Here $\det(\boldsymbol{V}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}_2-\lambda\boldsymbol{I})$ is a polynomial of degree of $n-q$. From the above equation we see that $\det(\boldsymbol{A}-\lambda\boldsymbol{I})$ has at least q repeated roots for $\lambda=\lambda_0$.

Secondly, the complex eigenvalues and eigenvectors could be proved by extending **orthogonal** matrix into **unitary** matrix.

The proof is complete. ∎



10 — Week10

10.1 Basic Theory for linear systems of ODEs

It is convenient to write the linear system of ODEs into compact matrix form:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{g}(t) \\ \boldsymbol{x}(t = t_0) = \boldsymbol{x}_0 \end{cases}$$

And it is meaning to consider the corresponding homogeneous equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{A}(t)\boldsymbol{x} \tag{10.1}$$

Once this equation has been solved, there are several methods that could be used to solve the nonhomogeneous equation, which will be introduced in the end of this chapter. But in this lecture, let's talk about some basic theory about the homogeneous equation:

Theorem 10.1 — Principle of superposition.
If the vector $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions to Eq.(10.1), then the linear combination $c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)}$ is also a solution for $c_1, c_2 \in \mathbb{R}$.

The proof is easy to verify. As a result, we could conclude that if $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(k)}$ are solutions of Eq.(10.1), then

$$\boldsymbol{x} = c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)} + \cdots + c_k\boldsymbol{x}^{(k)}$$

is also a solution to Eq.(10.1) for $c_i \in \mathbb{R}$.

10.1.1 Linear independent

Since any linear combinations is the solution to Eq.(10.1), we want to ask can all solution of Eq.(10.1) be expressed in this way?
The statement is true if $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$ are n **linearly independent** vectors:

Theorem 10.2 If $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are n **linearly independent** solutions to Eq.(10.1) for $t \in I$, where I is an open interval, then every solution $\boldsymbol{x} = \boldsymbol{\psi}(t)$ could be expressed as the unique linear combination of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$:

$$\boldsymbol{\psi}(t) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n$$

Proof. The result is obtained if there is a unique solution to the system of equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\psi}(t)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n \end{bmatrix}$$

Or equivalently, $\boldsymbol{A}$ is invertible. And we know that $\boldsymbol{A}$ is invertible if and only if all columns of $\boldsymbol{A}$ are **linearly independent**. Since $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are n **linearly independent** solutions, the proof is complete. ■

Definition 10.1 If $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n$ are **linearly** independent solutions to Eq.(10.1), then $\{\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n\}$ is said to form a **fundamental set of solutions** of Eq.(10.1). And the linear combination

$$\boldsymbol{x}_c = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n$$

is called the **general solution** of Eq.(10.1). ■

10.1.2 *Abel's Theorem*

But how to determine the linear independence of $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$? We only need to consider the determinant of the matrix $\boldsymbol{X}(t)$ whose columns are the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$:

Definition 10.2 — Wronskian. For solutions to system(10.1), $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, the **Wronskian** is given by:

$$W(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)(t) = \begin{vmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n \end{vmatrix}$$

The solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are **lienarly independent** at $t = t_0$ if and only if $W(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)(t)$ is **nonzero** at $t = t_0$. ■

But it is messy to determine the independence of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ on interval I if we have to determine whether the Wronskian is nonzero in the whole interval. We could simplify this problem by **Abel's theorem**:

Theorem 10.3 — Abel's theorem. For systems of ODE

$$\boldsymbol{x}' = \boldsymbol{A}(t)\boldsymbol{x}$$

with fundamental set of solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, the Wronskian formula is given by:

$$W(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)(t) = c \exp\left[\int \mathrm{tr}[\boldsymbol{A}(t)]\,\mathrm{d}t\right] \tag{10.2}$$

Proof. We set

$$\boldsymbol{X}(t) = \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \cdots & \boldsymbol{X}_{1n} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \cdots & \boldsymbol{X}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{n1} & \boldsymbol{X}_{n2} & \cdots & \boldsymbol{X}_{nn}\end{bmatrix}$$

- Firstly, we show that $\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \mathrm{tr}(\boldsymbol{A})W(t)$: We set

  $$W(t) = \det[\boldsymbol{X}(t)] = \sum_{i=1}^{n} C_{ij}\boldsymbol{X}_{ij}, \text{ where } C_{ij} \text{ is the corresponding cofactor.}$$

  It follows that

  $$\frac{\partial}{\partial \boldsymbol{X}_{ij}} W = \frac{\partial}{\partial \boldsymbol{X}_{ij}} \det[\boldsymbol{X}(t)] = \frac{\partial}{\partial \boldsymbol{X}_{ij}}\left(\sum_{i=1}^{n} C_{ij}\boldsymbol{X}_{ij}\right) = C_{ij}, \qquad i,j \in \{1,2,\ldots,n\}$$

  Thus

  $$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \frac{\mathrm{d}}{\mathrm{d}t}\det[\boldsymbol{X}(t)] = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial \boldsymbol{X}_{ij}}\det[\boldsymbol{X}(t)]\right]\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}_{ij}(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\boldsymbol{X}'_{ij}(t). \tag{10.3}$$

  Then we set

  $$\boldsymbol{C} = \begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{12} & \cdots & \boldsymbol{C}_{1n} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} & \cdots & \boldsymbol{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{n1} & \boldsymbol{C}_{n2} & \cdots & \boldsymbol{C}_{nn}\end{bmatrix}$$

  It follows that Eq.(10.3) could be expressed as:

  $$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \mathrm{tr}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{X}') \tag{10.4}$$

  Since $\boldsymbol{x}_i$ are solutions of (10.1), we obtain:

  $$\boldsymbol{x}'_i = \boldsymbol{A}\boldsymbol{x} \text{ for } i = 1,2,\ldots,n$$

  Or equivalently,

  $$\boldsymbol{X}' = \begin{bmatrix}\boldsymbol{x}'_1 & \boldsymbol{x}'_2 & \cdots & \boldsymbol{x}'_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}\boldsymbol{x}_1 & \boldsymbol{A}\boldsymbol{x}_2 & \cdots & \boldsymbol{A}\boldsymbol{x}_n\end{bmatrix} = \boldsymbol{A}\boldsymbol{X}. \tag{10.5}$$

  Plugging Eq.(10.5) into Eq.(10.4), we obtain:

  $$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \mathrm{tr}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{X}) = \mathrm{tr}\left[(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A})\cdot(\boldsymbol{X})\right]$$

  Since $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA})$ for arbitrary matrix $\boldsymbol{A}$ and $\boldsymbol{B}$, we derive:

  $$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \mathrm{tr}\left[(\boldsymbol{X})\cdot(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A})\right] = \mathrm{tr}\left[(\boldsymbol{X}\boldsymbol{C}^{\mathrm{T}})\cdot(\boldsymbol{A})\right]$$

By the definition of determinant, we find:

$$\boldsymbol{X}\boldsymbol{C}^{\mathrm{T}} = \begin{pmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \cdots & \boldsymbol{X}_{1n} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \cdots & \boldsymbol{X}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{n1} & \boldsymbol{X}_{n2} & \cdots & \boldsymbol{X}_{nn} \end{pmatrix} \begin{pmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{21} & \cdots & \boldsymbol{C}_{n1} \\ \boldsymbol{C}_{12} & \boldsymbol{C}_{22} & \cdots & \boldsymbol{C}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{1n} & \boldsymbol{C}_{2n} & \cdots & \boldsymbol{C}_{nn} \end{pmatrix} = \begin{pmatrix} \det(\boldsymbol{X}) & & \\ & \ddots & \\ & & \det(\boldsymbol{X}) \end{pmatrix} = \boldsymbol{I}\det(\boldsymbol{X})$$

Hence we derive:

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \mathrm{tr}[\boldsymbol{I}\cdot\det(\boldsymbol{X})\cdot\boldsymbol{A}(t)] = \mathrm{tr}[\boldsymbol{A}(t)]\det(\boldsymbol{X}) = \mathrm{tr}[\boldsymbol{A}(t)]W(t). \tag{10.6}$$

- It is easy to obtain the formula for $W(t)$ from the Eq.(10.6):

$$W(\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_n)(t) = c\exp\left[\int \mathrm{tr}[\boldsymbol{A}(t)]\,\mathrm{d}t\right].$$

■

R This theorem guarentees that either the Wronskian is zero for all $t \in I$ or else is never zero in I.

10.2 Linear ststem with constant coefficients (2×2 system)

We will focus on systems of **homogeneous linear** equations with **constant coefficients**, that is, the systems of the form

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} \tag{10.7}$$

where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is a constant matrix.
Let's focus on 2×2 system first:

2×2 ***system:***
Specifically speaking, we want to solve for the system:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned}$$

■ **Solution 10.1** Firstly, we talk about a case where $a_{21} = 0$, the system is given as:

$$x_1' = a_{11}x_1 + a_{12}x_2 \tag{10.8}$$
$$x_2' = \qquad a_{22}x_2 \tag{10.9}$$

By the Eq.(10.9), we derive:

$$x_2 = c_1 \exp(a_{22}t)$$

Substituting x_2 into the first equation, we obtain:

$$x_1' = a_{11}x_1 + c_1 a_{12} \exp(a_{22}t)$$

Solving for this first order ODE, we derive:

$$x_1 = \exp(a_{11}t)\left[\int \exp\{(a_{22} - a_{11})t\}\,\mathrm{d}t\right] = c_1\left(\frac{a_{12}}{a_{22} - a_{11}}\right)\exp(a_{22}t) + c_2 \exp(a_{11}t)$$

Hence we write the solution in vector valued form:

$$\boldsymbol{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} \frac{a_{12}}{a_{22}-a_{11}} \\ 1 \end{pmatrix} e^{a_{22}t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{a_{11}t}$$

If we set $r_1 = a_{22}, r_2 = a_{11}$, $\boldsymbol{\xi}^1 = \begin{pmatrix} \frac{a_{12}}{a_{22}-a_{11}} \\ 1 \end{pmatrix}, \boldsymbol{\xi}^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the result could be written as:

$$\boldsymbol{x} = c_1 \boldsymbol{\xi}^1 e^{r_1 t} + c_2 \boldsymbol{\xi}^2 e^{r_2 t}$$

■

R Concentrating on the result, we find that $(r_1, \boldsymbol{\xi}^1)$ and $(r_2, \boldsymbol{\xi}^2)$ are two eigen-pairs of $\boldsymbol{A}$. The solution is given by:

$$\boldsymbol{x} = c_1 \boldsymbol{\xi}^1 e^{r_1 t} + c_2 \boldsymbol{\xi}^2 e^{r_2 t}$$

■ **Solution 10.2** If $a_{21} \neq 0$, we want to solve for system

$$x_1' = a_{11}x_1 + a_{12}x_2 \tag{10.10}$$
$$x_2' = a_{21}x_1 + a_{22}x_2 \tag{10.11}$$

For Eq.(10.11), we solve for x_1 in terms of x_2:

$$x_1 = \frac{1}{a_{21}}(x_2' - a_{22}x_2). \tag{10.12}$$

Substituting x_1 into Eq.(10.10), we obtain:

$$\frac{1}{a_{21}}(x_2'' - a_{22}x_2') = a_{11} \cdot \frac{1}{a_{21}}(x_2' - a_{22}x_2) + a_{12}x_2$$

Or equivalently,

$$x_2'' - (a_{11} + a_{22})x_2' + (a_{11}a_{22} - a_{21}a_{12})x_2 = 0.$$

The characteristic equation is given by:

$$r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{21}a_{12}) = 0. \tag{10.13}$$

Suppose r_1 and r_2 are roots to this characteristic equation.

- For $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, we solve for $x_2(t)$ first:

  $$x_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

  By Eq.(10.12), we derive

  $$\begin{aligned} x_1(t) &= \frac{1}{a_{21}}(c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} - c_1 a_{22} e^{r_1 t} - c_2 a_{22} e^{r_2 t}) \\ &= c_1 \left(\frac{r_1 - a_{22}}{a_{21}}\right) e^{r_1 t} + c_2 \left(\frac{r_2 - a_{22}}{a_{21}}\right) e^{r_2 t} \end{aligned}$$

  Hence we write the solution in vector-valued form:

  $$\boldsymbol{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \boldsymbol{\xi}^1 e^{r_1 t} + c_2 \boldsymbol{\xi}^2 e^{r_2 t}.$$

  where

  $$\boldsymbol{\xi}^1 = \begin{pmatrix} \frac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix}, \qquad \boldsymbol{\xi}^2 = \begin{pmatrix} \frac{r_2 - a_{22}}{a_{21}} \\ 1 \end{pmatrix}$$

  Notice that $(r_1, \boldsymbol{\xi}^1)$ and $(r_2, \boldsymbol{\xi}^2)$ are two eigen-pairs of $\boldsymbol{A}$.

- For $r_1, r_2 \in \mathbb{C}$, i.e. $r_1 = \bar{r_2}$, we suppose $r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu$.
  We write one of the fundamental solution in previous case into complex-valued form:

$$\boldsymbol{\xi}^1 e^{r_1 t} = \begin{pmatrix} \frac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{r_1 t} = \begin{pmatrix} \frac{\lambda + i\mu - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{\lambda t}(\cos\mu t + i\sin\mu t)$$

$$= \left[\begin{pmatrix} \frac{\lambda - a_{22}}{a_{21}} \\ 1 \end{pmatrix}\cos\mu t - \begin{pmatrix} \frac{\mu}{a_{21}} \\ 0 \end{pmatrix}\sin\mu t\right] e^{\lambda t} + i\left[\begin{pmatrix} \frac{\lambda - a_{22}}{a_{21}} \\ 1 \end{pmatrix}\sin\mu t + \begin{pmatrix} \frac{\mu}{a_{21}} \\ 0 \end{pmatrix}\cos\mu t\right] e^{\lambda t}$$

  We take real and imaginary parts as two **linearly independent** solutions, then we could obtain the real-valued solutions:

$$\boldsymbol{x} = c_1\left[\begin{pmatrix} \frac{\lambda - a_{22}}{a_{21}} \\ 1 \end{pmatrix}\cos\mu t - \begin{pmatrix} \frac{\mu}{a_{21}} \\ 0 \end{pmatrix}\sin\mu t\right] e^{\lambda t} + c_2\left[\begin{pmatrix} \frac{\lambda - a_{22}}{a_{21}} \\ 1 \end{pmatrix}\sin\mu t + \begin{pmatrix} \frac{\mu}{a_{21}} \\ 0 \end{pmatrix}\cos\mu t\right] e^{\lambda t}.$$

- For $r_1 = r_2 = r \in \mathbb{R}$, due to the characteristic equation(10.13), we could solve for $x_2(t)$:

$$x_2(t) = c_1 e^{rt} + c_2 t e^{rt}$$

  By Eq.(10.12), we derive

$$x_1(t) = \frac{1}{a_{21}}(c_1 r e^{rt} + c_2 e^{rt} + c_2 r t e^{rt} - c_1 a_{22} e^{rt} - c_2 a_{22} t e^{rt})$$
$$= c_1 \frac{r - a_{22}}{a_{21}} e^{rt} + c_2 \frac{r - a_{22}}{a_{21}} t e^{rt} + c_2 \frac{1}{a_{21}} e^{rt}.$$

  Hence we write the solution in vector-valued form:

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} \frac{r - a_{22}}{a_{21}} \\ 1 \end{pmatrix} e^{rt} + c_2\left[\begin{pmatrix} \frac{r - a_{22}}{a_{21}} \\ 1 \end{pmatrix} t + \begin{pmatrix} \frac{1}{a_{21}} \\ 0 \end{pmatrix}\right] e^{rt}$$

■

Then we summarize how to solve the 2×2 system of ODE:

R

- For $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$,
  $AM = GM$ (*algebraic multiplicity=geometric multiplicity*). Suppose $(r_1, \boldsymbol{\xi}^1)$ and $(r_2, \boldsymbol{\xi}^2)$ are two eigen-pairs of $\boldsymbol{A}$. The solution is given by:
  $$x(t) = c_1 \boldsymbol{\xi}^1 e^{r_1 t} + c_2 \boldsymbol{\xi}^2 e^{r_2 t}.$$
- For $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$,
  $AM = GM$. We need to take **real** and **imaginary** parts of $\boldsymbol{\xi}^1 e^{r_1} t$ or $\boldsymbol{\xi}^2 e^{r_2} t$, as another pair of two **linearly independent** solutions.
- For $r_1 = r_2 = r \in \mathbb{R}$,
  r is eigenvalue of $\boldsymbol{A}$ with $AM = 2$. Suppose $\boldsymbol{\xi} = \begin{pmatrix} \frac{r - a_{22}}{a_{21}} \\ 1 \end{pmatrix}$ is the eigenvector associated with r, $GM = 1$.
  The *first* solution is $\boldsymbol{\xi} e^{rt}$, the *second* solution is not $\xi t e^{rt}$! but to be $(\boldsymbol{\xi} t + \boldsymbol{\eta}) e^{rt}$, where $\boldsymbol{\eta}$ is **generalized eigenvector** such that
  $$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi} \implies \eta = \begin{pmatrix} \frac{1}{a_{21}} \\ 0 \end{pmatrix}$$
  Further more, the more generalized eigenvector is ζ s.t.
  $$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\zeta}$$

■ **Example 10.1** Solve $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \boldsymbol{x}$:

For $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, the eigenvalues satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = 0 = \begin{vmatrix} 1-r & 1 \\ -1 & 3-r \end{vmatrix} = r^2 - 4r + 4. \implies r_1 = r_2 = 2.$$

For $r = 2$, the corresponding eigenvector $\boldsymbol{\xi}$ satisfies

$$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\xi} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \boldsymbol{\xi} = \boldsymbol{0} \implies \boldsymbol{\xi} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence the first solution is given by:

$$\boldsymbol{\xi} e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The **generalized eigenvector** η associated with $r = 2$ satisfies

$$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi} \implies \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \boldsymbol{\eta} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence the second solution is given by:

$$(\boldsymbol{\xi} t + \boldsymbol{\eta}) e^{rt} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] e^{2t}.$$

Hence the general solution is given by:

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] e^{2t}.$$

■

■ **Example 10.2** Solve $\boldsymbol{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \boldsymbol{x}$:

For $\boldsymbol{A} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$, the eigenvalues satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = 0 = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}. \implies r_1 = -\frac{1}{2} + i, \quad r_2 = -\frac{1}{2} - i.$$

We find the corresponding eigenvector of r_1:

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi}^1 = \boldsymbol{0} \implies \begin{vmatrix} -i & 1 \\ -1 & -i \end{vmatrix} \boldsymbol{\xi}^1 = \boldsymbol{0} \implies \boldsymbol{\xi}^1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The first solution is given by:

$$\boldsymbol{x}^{(1)}(t)=\boldsymbol{\xi}^{1}e^{r_1 t}=\begin{pmatrix}1\\ i\end{pmatrix}e^{(-\frac{1}{2}+i)t}=\begin{pmatrix}1\\ i\end{pmatrix}e^{-t/2}(\cos t+i\sin t)=\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}e^{-t/2}+i\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}e^{-t/2}$$

We take the real and imaginary part of $\boldsymbol{x}^{(1)}$ to form two linearly independent fundamental set of solutions:

$$\boldsymbol{u}=\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}e^{-t/2},\quad \boldsymbol{v}=\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}e^{-t/2}.$$

Thus the general solution is given by:

$$\boldsymbol{x}=c_1\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}e^{-t/2}+c_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}e^{-t/2}.$$

■

11 — Week11

11.1 Linear system with constant coefficients (3×3 system)

Now we have learnt how to solve a 2×2 linear system with constant coefficients. Let's talk about the 3×3 case:

$$\boldsymbol{x}' = \boldsymbol{A}_{3\times 3}\boldsymbol{x} \tag{11.1}$$

But notice that in this lecture we will **not** talk about the case where $\boldsymbol{A}$ has **three** repeated eigenvalue in Eq.(11.1). This is because in the section, we will introduce a much easier and much beautiful technique to solve this case.

11.1.1 *Matrix A has real, distinct eigenvalues*

In this case, we just need to find three eigen-pairs $(r_i, \boldsymbol{\xi}^i)$ for $i = 1,2,3$. The general solution is given by:

$$\boldsymbol{x} = c_1\boldsymbol{\xi}^1 e^{r_1 t} + c_2\boldsymbol{\xi}^2 e^{r_2 t} + c_3\boldsymbol{\xi}^3 e^{r_3 t}$$

■ **Example 11.1** Solve $\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The eigenvalues of $\boldsymbol{A}$ must satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{vmatrix} = -(r+1)(r-1)(r-2) = 0 \implies r_1 = 2, \quad r_2 = -1, \quad r_3 = 1.$$

The eigenvector corresponding to $r_1 = 2$ must satisfy:

$$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\xi}^1 = \boldsymbol{0} \implies \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \boldsymbol{\xi}^1 = \boldsymbol{0} \implies \boldsymbol{\xi}^1 = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We could find another two eigen-pairs $(r_2, \boldsymbol{\xi}^2)$ and $(r_3, \boldsymbol{\xi}^3)$ similarly, where

$$\boldsymbol{\xi}^2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{\xi}^3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Thus the general solution is:

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

■

11.1.2 Matrix A has complex eigenvalues

In this case we have two complex eigen-pairs $(r_1, \boldsymbol{\xi}^1)$,$(r_2, \boldsymbol{\xi}^2)$ and one real eigen-pair $(r_3, \boldsymbol{\xi}^3)$. One solution to system(11.1) is

$$\boldsymbol{\xi}^3 e^{r_3 t}$$

In order to obtain another two real-valued solutions, we need to take the **real** and **complex** part of complex-valued solution $\boldsymbol{\xi}^1 e^{r_1 t}$ or $\boldsymbol{\xi}^2 e^{r_2 t}$.

■ **Example 11.2** Solve $\boldsymbol{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$.

The eigenvalues of $\boldsymbol{A}$ must satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = (r-1)(r^2 - 2r + 5) = 0 \implies r_1 = 1, \quad r_2 = 1 + 2i, \quad r_3 = 1 - 2i.$$

The eigenvector corresponding to $r_1 = 1$ must satisfy:

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \boldsymbol{\xi}^1 = \boldsymbol{0} \implies \boldsymbol{\xi}^1 = c \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

The eigenvector corresponding to $r_2 = 1 + 2i$ must satisfy:

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} \boldsymbol{\xi}^2 = \boldsymbol{0} \implies \boldsymbol{\xi}^2 = c \begin{pmatrix} 0 \\ i \\ i \end{pmatrix}$$

We select the **real** and **imaginary** part of $\boldsymbol{\xi}^2$:

$$\boldsymbol{x}^{(2)} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} e^t(\cos 2t + i\sin 2t) = \begin{pmatrix} 0 \\ -e^t\sin 2t \\ e^t\cos 2t \end{pmatrix} + i\begin{pmatrix} 0 \\ e^t\cos 2t \\ e^t\sin 2t \end{pmatrix}$$

Thus the general solution is given by:

$$\boldsymbol{x} = c_1\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2\begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} e^t + c_3\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} e^t.$$

11.1.3 Matrix A has two repeated eigenvalues

In this case we have two real eigen-pairs $(r_1,\boldsymbol{\xi}^1)$ and $(r_2,\boldsymbol{\xi}^2)$, where r_1 is the eigenvalue of $\boldsymbol{A}$ with multiplicity 2.
Two solutions to system(11.1) is:

$$\boldsymbol{\xi}^1 e^{r_1 t}, \qquad \boldsymbol{\xi}^2 e^{r_2 t}$$

In order to obtain another one solution, firstly we should find a **generalized eigenvector** $\boldsymbol{\eta}$ by solving the equation

$$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}^1$$

Then the third solution is given by:

$$\boldsymbol{x}^{(3)} = (t\boldsymbol{\xi}^1 + \boldsymbol{\eta})e^{r_1 t}$$

Hence the general solution is given by:

$$\boldsymbol{x} = c_1\boldsymbol{\xi}^1 e^{r_1 t} + c_2\boldsymbol{\xi}^2 e^{r_2 t} + c_3(t\boldsymbol{\xi}^1 + \boldsymbol{\eta})e^{r_1 t}$$

■ **Example 11.3** Solve $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\boldsymbol{x}$.

The eigenvalue of $\boldsymbol{A}$ must satisfy

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = -(r+1)(r-2)^2 = 0 \implies r_1 = -1, r_2 = r_3 = 2.$$

The eigenvector corresponding to $r_1 = -1$ must satisfy:

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\boldsymbol{\xi}^{(1)} = \boldsymbol{0} \implies \boldsymbol{\xi}^{(1)} = c\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$$

The eigenvector corresponding to $r_2 = 2$ must satisfy:

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\boldsymbol{\xi}^{(2)} = \boldsymbol{0} \implies \boldsymbol{\xi}^{(2)} = c\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The generalized eigenvector of $r_2 = 2$ must satisfy:

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \boldsymbol{\eta} = \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \implies \boldsymbol{\eta} = c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

In conclusion, the general solution is

$$\begin{aligned} \boldsymbol{x} &= c_1 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right] + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} \\ &= c_1 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ t+1 \\ -t \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}. \end{aligned}$$

■

11.2 Fundamental solution matrix

This lecture we will study the solutions of systems of ODE in detail.

Definition 11.1 Suppose that $\boldsymbol{x}^{(1)}(t),\dots,\boldsymbol{x}^{(n)}(t)$ form a fundamental set of solutions for the equation

$$\boldsymbol{X}' = \boldsymbol{P}(t)\boldsymbol{X} \tag{11.2}$$

on open interval I. Then

$$\Psi(t) = \begin{bmatrix}\boldsymbol{x}^{(1)}(t) & \boldsymbol{x}^{(2)}(t) & \dots & \boldsymbol{x}^{(n)}(t)\end{bmatrix}$$

is called the **fundamental matrix** for system(11.2). Note that a fundamental matrix is **nonsingular** since its columns are **linearly independent**. ■

Our question is that how to use fundamental matrix to solve IVP

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} \tag{11.3}$$

$$\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0 \tag{11.4}$$

■ **Solution 11.1** The general solution to Eq.(11.3) is given by:

$$\boldsymbol{x} = \boldsymbol{c}_1\boldsymbol{x}^{(1)}(t) + \dots + c_n\boldsymbol{x}^{(n)}(t)$$

Or equivalently,

$$\boldsymbol{x} = \Psi(t)\boldsymbol{c} \tag{11.5}$$

where $\boldsymbol{c} = (c_1, c_2, \dots, c_n)^{\mathrm{T}}$.
Since $\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0$, the Eq.(11.5) should satisfy:

$$\Psi(t = t_0)\boldsymbol{c} = \boldsymbol{x}_0$$

Since $\Psi(t_0)$ is nonsingular, we derive:

$$\boldsymbol{c} = \Psi^{-1}(t_0)\boldsymbol{x}_0$$

The solution to the IVP(11.3) and (11.4) is given by:

$$\boldsymbol{x} = \Psi(t)\Psi^{-1}(t_0)\boldsymbol{x}_0 \tag{11.6}$$

■

Next, we want to construct another fundamental matrix $\phi(t)$ such that $\phi(t_0) = \boldsymbol{I}$. Let's construct it by using $\phi(t)$ to derive the solution to the previous IVP:

■ **Solution 11.2** Since $\phi(t)$ is a fundamental solution matrix to Eq.(11.3), the solution to the IVP (11.3) and (11.4) could be expressed as:

$$\boldsymbol{x} = \phi(t)\phi^{-1}(t_0)\boldsymbol{x}_0$$

Since $\phi(t_0) = \boldsymbol{I}$, we find $\phi^{-1}(t_0) = \boldsymbol{I}$. Thus the solution is given as:

$$\boldsymbol{x} = \phi(t)\boldsymbol{x}_0 \tag{11.7}$$

Comparing Eq.(11.6) and Eq.(11.7), the matrix $\phi(t)$ must satisfy:

$$\phi(t) = \Psi(t)\Psi^{-1}(t_0). \tag{11.8}$$

■

■ **Example 11.4** For $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{x}$, we want to find the fundamental matrix $\phi(t)$ s.t. $\phi(t_0) = \boldsymbol{I}$.
We have two eigenpairs $(r_1, \boldsymbol{\xi}_1)$ and $(r_2, \boldsymbol{\xi}_2)$, where

$$r_1 = 3, \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = -1, \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Then we find

$$\boldsymbol{x}^{(1)}(t) = \boldsymbol{\xi}_1 e^{r_1 t}, \quad \boldsymbol{x}^{(2)}(t) = \boldsymbol{\xi}_2 e^{r_2 t}$$

Thus we obtain the fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -2e^{-t} \end{bmatrix}$$

It follows that

$$\phi(t) = \Psi(t)\Psi^{-1}(t_0).$$

■

For linear system ODE with constant coefficients, we want to find a matrix $\phi(t)$ s.t.

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = \boldsymbol{A}\phi(t)$$
$$\phi(t=0) = \boldsymbol{I}.$$

- Case 1: for $n = 1$, we have

  $$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = a\phi(t)$$
  $$\phi(0) = 1.$$

  The solution is $\phi(t) = e^{at}$.
- Case 2: for $n > 1$, is $\phi(t) = e^{\boldsymbol{A}t}$?
  Our first question is how to define $e^{\boldsymbol{A}t}$?
  Recall that for $a \in \mathbb{C}$,

  $$\begin{aligned} e^{at} &= 1 + at + \frac{(at)^2}{2!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(at)^n}{n!} \end{aligned}$$

Definition 11.2 — Exponential of matrix. If $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the exponential of $\boldsymbol{A}t$ is given by:

$$
\begin{aligned}
e^{\boldsymbol{A}t} &= \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}t)^n}{n!} \\
&= \boldsymbol{I} + t\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \dots
\end{aligned}
$$

Let's show that $e^{\boldsymbol{A}t}$ is well-defined first: (the related definitions and theorems are listed below)

Definition 11.3 — Matrix Norm. The **norm** of a matrix $\boldsymbol{A}$ is the number

$$||\boldsymbol{A}|| = \sup_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{||\boldsymbol{A}\boldsymbol{x}||}{||\boldsymbol{x}||}$$

Or equivalently,

$$||\boldsymbol{A}|| = \sup_{||\boldsymbol{x}||=1} ||\boldsymbol{A}\boldsymbol{x}||$$

the **sphere** $||\boldsymbol{A}\boldsymbol{x}||$ for $||\boldsymbol{x}|| = 1$ is compact, and $||\boldsymbol{A}\boldsymbol{x}||$ is continuous, thus $0 < ||\boldsymbol{A}|| < \infty$.

Theorem 11.1 — The Weierstrass Criterion. For a series $\sum_{i=1}^{\infty} f_i$ of functions $f_i : X \mapsto M$, if we have

$$||f_i|| \le a_i, \qquad \sum_{i=1}^{\infty} a_i \text{ is convergent}, \qquad a_i \in \mathbb{R},$$

then $\sum_{i=1}^{\infty} f_i$ is **uniformly convergent**.

Let's show the validity of $e^{\boldsymbol{A}t}$ by showing the uniform convergence of it:

Theorem 11.2 For an open interval I, the series $e^{\boldsymbol{A}t}$ uniformly converges for any $\boldsymbol{A}$ and any $t \in I$.

Proof. Since t is bounded, there exists $a \in \mathbb{R}$ such that

$$||\boldsymbol{A}t|| \le a.$$

Since the numerical series $\sum_{n=0}^{\infty} \frac{a^n}{n!}$ converges to e^a, by *Weierstrass criterion theorem*, the exponential $e^{\boldsymbol{A}t}$ is uniformly convergent for $||\boldsymbol{A}t|| \le a$. ∎

After showing $e^{\boldsymbol{A}t}$ is well-defined, we could show that $e^{\boldsymbol{A}t}$ is exactly $\phi(t)$:

Let $\phi(t) = e^{\mathbf{A}t}$, then we obtain:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=0}^{\infty}\frac{(\mathbf{A}t)^n}{n!}\right] = \sum_{n=0}^{\infty}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{(\mathbf{A}t)^n}{n!}\right\} \\
&= 0 + \mathbf{A} + t\mathbf{A}^2 + \dots \\
&= \mathbf{A}\left(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots\right) \\
&= \mathbf{A}e^{\mathbf{A}t} = \mathbf{A}\phi(t).
\end{aligned}
$$

Also, for $\phi(t=0)$, we derive

$$
\phi(t=0) = \sum_{n=0}^{\infty}\frac{(\mathbf{A}\cdot 0)^n}{n!} = \mathbf{I}.
$$

Note that the derivative and summation sign could be exchanged due to the uniform convergence of the series.

Then we show some interesting properties of $e^{\mathbf{A}}$:

Proposition 11.1
1. $\exp(\mathbf{0}_{n\times n}) = \mathbf{I}$.
2. If $\mathbf{AB} = \mathbf{BA}$, then $\exp(\mathbf{A}+\mathbf{B}) = e^{\mathbf{A}}e^{\mathbf{B}}$.
3. $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$
4. For non-singular matrix $\mathbf{P}$, we have

$$
\exp(\mathbf{PAP}^{-1}) = \mathbf{P}e^{\mathbf{A}}\mathbf{P}^{-1}.
$$

Proof. For (2), we have

$$
\exp(\mathbf{A}+\mathbf{B}) = \sum_{n=0}^{\infty}\frac{(\mathbf{A}+\mathbf{B})^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}\mathbf{A}^j\mathbf{B}^{n-j} \quad \text{By condition } \mathbf{AB} = \mathbf{BA}.
$$

It follows that

$$
\begin{aligned}
\exp(\mathbf{A}+\mathbf{B}) &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\mathbf{A}^j\mathbf{B}^{n-j} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\mathbf{A}^j\mathbf{B}^{n-j} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{\mathbf{A}^j}{j!}\frac{\mathbf{B}^{n-j}}{(n-j)!} \\
&= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{\mathbf{A}^j}{j!}\frac{\mathbf{B}^{n-j}}{(n-j)!} \\
&= \sum_{j=0}^{\infty}\frac{\mathbf{A}^j}{j!}\left[\sum_{n=j}^{\infty}\frac{\mathbf{B}^{n-j}}{(n-j)!}\right] \\
&= \sum_{j=0}^{\infty}\frac{\mathbf{A}^j}{j!}e^{\mathbf{B}} \\
&= e^{\mathbf{A}}e^{\mathbf{B}}.
\end{aligned}
$$

For (3), we only need to show $e^{\mathbf{A}}e^{-\mathbf{A}} = \mathbf{I}$. Since

$$
e^{\mathbf{A}}(e^{\mathbf{A}})^{-1} = \mathbf{I} = \exp(\mathbf{0}_{n\times n}) = e^{\mathbf{A}-\mathbf{A}} = e^{\mathbf{A}}e^{-\mathbf{A}}. \text{ since } \mathbf{A}(-\mathbf{A}) = (-\mathbf{A})\mathbf{A}.
$$

For (4), we observe

$$
\begin{aligned}
\exp(\boldsymbol{PAP}^{-1}) &= \sum_{n=0}^{\infty} \frac{(\boldsymbol{PAP}^{-1})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\boldsymbol{PA}^n\boldsymbol{P}^{-1}}{n!} \\
&= \boldsymbol{P}\left[\sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n}{n!}\right] \\
&= \boldsymbol{P}e^{\boldsymbol{A}}\boldsymbol{P}^{-1}
\end{aligned}
$$

■

By this proposition, we derive:

- $\phi(t+s) = \phi(t)\phi(s)$.
$$e^{\boldsymbol{A}(t+s)} = e^{\boldsymbol{A}t+\boldsymbol{A}s} = e^{\boldsymbol{A}t} \cdot e^{\boldsymbol{A}s}.$$
- $\phi(t-s) = \phi(t)\phi^{-1}(s)$.
$$e^{\boldsymbol{A}(t-s)} = e^{\boldsymbol{A}t}e^{-\boldsymbol{A}s} = e^{\boldsymbol{A}t}(e^{\boldsymbol{A}s})^{-1}.$$

Since the fundamental matrix could be computed as $e^{\boldsymbol{A}t}$, our question is that is there any efficient way to compute $e^{\boldsymbol{A}t}$? We will talk about it in the next lecture.

12 — Week12

12.1 Matrix Factorization

We want to find a fundamental solution matrix $\phi(t)$ such that $\phi(0) = \boldsymbol{I}$.
The question in this lecture is how to compute $e^{\boldsymbol{A}t}$, i.e.

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{t^2}{2}\boldsymbol{A}^2 + \cdots = \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}t)^n}{n!}.$$

In this lecture we only focus on the 2×2 and 3×3 matrix. Let's talk about three cases for computing the exponential of $\boldsymbol{A}t$:

12.1.1 *A is a diagonal matrix*

If $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ could be written as

$$\boldsymbol{A} = \operatorname{diag}(a_1, a_2, \ldots, a_m)$$

It follows that

$$\boldsymbol{A}^k = \operatorname{diag}(a_1^k, a_2^k, \ldots, a_m^k) \text{ for } k \in \mathbb{N}^+.$$

Thus the exponential of $\boldsymbol{A}t$ is given by:

$$\begin{aligned} e^{\boldsymbol{A}t} &= \sum_{k=0}^{\infty} \frac{(\boldsymbol{A}t)^k}{k!} \\ &= \operatorname{diag}\left(\sum_{k=0}^{\infty} \frac{(a_1 t)^k}{k!}, \sum_{k=0}^{\infty} \frac{(a_2 t)^k}{k!}, \ldots, \sum_{k=0}^{\infty} \frac{(a_m t)^k}{k!}\right) \\ &= \operatorname{diag}\left(e^{a_1 t}, e^{a_2 t}, \ldots, e^{a_m t}\right). \end{aligned}$$

■ **Example 12.1** Find the fundamental solution matrix and the general solution for system

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \boldsymbol{x}$$

The fundamental solution matrix is given by:

$$\phi(t) = e^{\boldsymbol{A}t} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

The general solution is given by:

$$\boldsymbol{x} = c_1 \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{2t} \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$$

■

12.1.2 *A is diagonalizable*

$\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is diagonalizable if and only if it has n **linearly independent** eigenvectors.

- Let me show you how to diagonalize $\boldsymbol{A}$ (if we can):
  Suppose $(r_1, \boldsymbol{\xi}^{(1)}), \dots, (r_n, \boldsymbol{\xi}^{(n)})$ are n eigen-pairs of $\boldsymbol{A}$. (since we have assumed $\boldsymbol{A}$ has n independent eigenvectors)
  It follows that

  $$\boldsymbol{A}\boldsymbol{\xi}^{(i)} = r_i \boldsymbol{\xi}^{(i)} \text{ for } i = 1, 2, \dots, n.$$

  Hence we derive:

  $$\boldsymbol{A}\begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \cdots & \boldsymbol{\xi}^{(n)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \cdots & \boldsymbol{\xi}^{(n)} \end{bmatrix} \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}$$

  If we set $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \cdots & \boldsymbol{\xi}^{(n)} \end{bmatrix}$, $\boldsymbol{\Lambda} = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}$, we obtain:

  $$\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}$$

  Sinc $\boldsymbol{V}$ has n independent columns, we find $\boldsymbol{V}$ is invertible. Hence we obtain the diagonalization of $\boldsymbol{A}$:

  $$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$$

- Once $\boldsymbol{A}$ is diagonalizable, we could compute $e^{\boldsymbol{A}t}$. Recall that for any matrix $\boldsymbol{Q}$,

  $$\exp(\boldsymbol{P}^{-1}\boldsymbol{Q}\boldsymbol{P}) = \boldsymbol{P}^{-1}e^{\boldsymbol{Q}}\boldsymbol{P}.$$

  Hence we obtain:

  $$e^{\boldsymbol{A}t} = e^{(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})t} = \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1}$$

where

$$e^{\boldsymbol{\Lambda}t} = \begin{pmatrix} e^{r_1 t} & & \\ & \ddots & \\ & & e^{r_n t} \end{pmatrix}$$

Actually, if you don't want to get $e^{\boldsymbol{A}t}$ (the fundamental solution matrix $\phi(t)$ satisfying $\phi(t) = \boldsymbol{I}$), if you only want to get a common fundamental solution matrix $\Psi(t)$, you can verify that $\Psi(t) = \boldsymbol{V}e^{\lambda t}$ is also a fundamental solution matrix.

■ **Example 12.2** Find the fundamental solution matrix $\phi(t)$ satisfying $\phi(0) = \boldsymbol{I}$ and the general solution for system

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \boldsymbol{x}$$

You can verify that the three eign-pairs of $\boldsymbol{A}$ is $(r_1, \boldsymbol{\xi}^{(1)}), (r_2, \boldsymbol{\xi}^{(2)}), (r_3, \boldsymbol{\xi}^{(3)})$, where

$$r_1 = 2, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \qquad r_2 = -1, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \qquad r_3 = -1, \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix};$$

Hence we set

$$\boldsymbol{\Lambda} = \operatorname{diag}(r_1, r_2, r_3), \qquad \boldsymbol{V} = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \boldsymbol{\xi}^{(3)} \end{bmatrix}$$

One fundamental solution matrix of this system is given by:

$$\Psi(t) = \boldsymbol{V}e^{\Lambda t} = \begin{bmatrix} \boldsymbol{\xi}^{(1)} e^{r_1 t} & \boldsymbol{\xi}^{(2)} e^{r_2 t} & \boldsymbol{\xi}^{(3)} e^{r_3 t} \end{bmatrix}$$

For fundamental solution matrix satisfying $\phi(t) = \boldsymbol{I}$, we have:

$$\begin{aligned}
\phi(t) = e^{\boldsymbol{A}t} = \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1} &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \end{pmatrix}.
\end{aligned}$$

■

12.1.3 A has only one distinct eigenvalue

In this part we introduce the $S - N$ decomposition:

Theorem 12.1 — *S*-*N* decomposition. Let $\boldsymbol{A}$ be a 2×2 or 3×3 matrix which has only one distinct eigenvalue r. Then $\boldsymbol{A}$ could be decomposed as $\boldsymbol{A}=\boldsymbol{S}+\boldsymbol{N}$ such that

1. $\boldsymbol{S}=r\boldsymbol{I}$.
2. $\boldsymbol{N}=\boldsymbol{A}-\boldsymbol{S}$.
3. $\boldsymbol{N}^2=\boldsymbol{0}$ or $\boldsymbol{N}^3=\boldsymbol{0}$.
4. $\boldsymbol{SN}=\boldsymbol{NS}$

We skip the proof for this theorem, if you are interested in this kind of decomposition (which is much easier than Jordan form), you could check the book "**Basic Theory of Ordinary Differential Equations**" (shown in Figure(12.1))

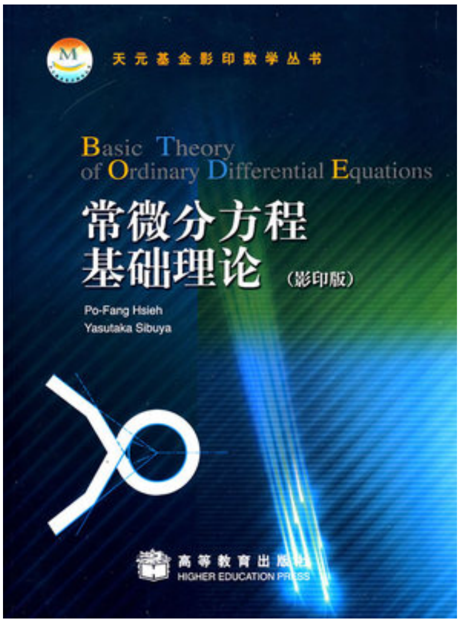


Figure 12.1: The book for *S*-*N* decomposition in detail

If $\boldsymbol{A}$ admits *S*-*N* decomposition, let's show how to compute the fundamental matrix $e^{\boldsymbol{A}t}$:

■ **Solution 12.1** For $\boldsymbol{A}\in\mathbb{C}^{n\times n}$ with eigenvalues r, we decompose it as

$$\boldsymbol{A}=\boldsymbol{S}+\boldsymbol{N},$$

where $\boldsymbol{S}=r\boldsymbol{I}$, $\boldsymbol{N}=\boldsymbol{A}-\boldsymbol{S}$, $\boldsymbol{N}^2=\boldsymbol{0}$ or $\boldsymbol{N}^3=\boldsymbol{0}$.
It follows that

$$e^{\boldsymbol{A}t}=e^{(\boldsymbol{S}+\boldsymbol{N})t}=e^{\boldsymbol{S}t+\boldsymbol{N}t}.$$

Since $\boldsymbol{SN}=\boldsymbol{NS}$, we derive:

$$e^{\boldsymbol{A}t}=e^{\boldsymbol{S}t}e^{\boldsymbol{N}t}$$

where

$$e^{\boldsymbol{S}t} = e^{\boldsymbol{I}(rt)} = \text{diag}(\underbrace{e^{rt}, e^{rt}, \ldots, e^{rt}}_{n \ \ terms}) = e^{rt}\boldsymbol{I}.$$

and

$$e^{\boldsymbol{N}t} = \sum_{k=0}^{\infty} \frac{(\boldsymbol{N}t)^k}{k!} = \sum_{k=0}^{2} \frac{(\boldsymbol{N}t)^k}{k!} = \boldsymbol{I} + \boldsymbol{N}t + \frac{1}{2}\boldsymbol{N}^2 t^2.$$

Hence the exponential $e^{\boldsymbol{A}t}$ could be computed as:

$$e^{\boldsymbol{A}t} = e^{rt}\left[\boldsymbol{I} + \boldsymbol{N}t + \frac{1}{2}\boldsymbol{N}^2 t^2\right].$$

■

■ **Example 12.3** Find a fundamental solution matrix of

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \boldsymbol{x}.$$

The eigenvalues of $\boldsymbol{A}$ must satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-2)^2 \implies r = 2.$$

We perform the S-N decompositon for $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{N} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

And we observe

$$e^{\boldsymbol{S}t} = e^{2t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\boldsymbol{N}^2 = \boldsymbol{0} \implies e^{\boldsymbol{N}t} = \boldsymbol{I} + \boldsymbol{N}t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & -t \\ t & t \end{pmatrix} = \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}$$

The fundamental matrix is given by:

$$e^{\boldsymbol{A}t} = e^{\boldsymbol{S}t}e^{\boldsymbol{N}t} = e^{2t}\begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}.$$

The general solution is given by:

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1-t \\ t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -t \\ 1+t \end{pmatrix} e^{2t}.$$

■

■ **Example 12.4** Find a fundamental solution matrix of

$$\boldsymbol{x}' = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{A}$ must satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{vmatrix} = -(r-1)^3 \implies r = 1.$$

We perform the S-N decompositon for $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}$$

And we observe

$$e^{\boldsymbol{S}t} = e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\boldsymbol{N}^2 = \boldsymbol{0} \implies e^{\boldsymbol{N}t} = \boldsymbol{I} + \boldsymbol{N}t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4t & -3t & -2t \\ 8t & -6t & -4t \\ -4t & 3t & 2t \end{pmatrix} = \begin{pmatrix} 4t+1 & -3t & -2t \\ 8t & -6t+1 & -4t \\ -4t & 3t & 2t+1 \end{pmatrix}$$

Thus the fundamental solution matrix is given by:

$$\boldsymbol{x} = c_1 \begin{pmatrix} 4t+1 \\ 8t \\ -4t \end{pmatrix} e^t + c_2 \begin{pmatrix} -3t \\ -6t+1 \\ 3t \end{pmatrix} e^t + c_3 \begin{pmatrix} -2t \\ -4t \\ 2t+1 \end{pmatrix} e^t.$$

■

■ **Example 12.5** Find a fundamental solution matrix of

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{A}$ must satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{vmatrix} = -(r-2)^3 \implies r = 2.$$

We perform the S-N decompositon for $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{N} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$$

And we observe

$$e^{\boldsymbol{S}t} = e^{2t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\boldsymbol{N}^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \boldsymbol{N}^3 = \boldsymbol{0}$$

It follows that

$$e^{\boldsymbol{N}t} = \boldsymbol{I} + \boldsymbol{N}t + \frac{1}{2}\boldsymbol{N}^2 t^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & t & t \\ 2t & -t & -t \\ -3t & 2t & 2t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -\frac{t^2}{2} & \frac{t^2}{2} & \frac{t^2}{2} \\ \frac{t^2}{2} & -\frac{t^2}{2} & -\frac{t^2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1-t & t & t \\ 2t - \frac{t^2}{2} & 1 - t + \frac{t^2}{2} & -t + \frac{t^2}{2} \\ -3t + \frac{t^2}{2} & 2t - \frac{t^2}{2} & 1 + 2t - \frac{t^2}{2} \end{pmatrix}.$$

Thus the fundamental solution matrix is given by:

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1-t \\ 2t - \frac{t^2}{2} \\ -3t + \frac{t^2}{2} \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t \\ 1 - t + \frac{t^2}{2} \\ 2t - \frac{t^2}{2} \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} t \\ -t + \frac{t^2}{2} \\ 1 + 2t - \frac{t^2}{2} \end{pmatrix} e^{2t}.$$

■

12.1.4 Nonhomogeneous linear system

Finally, we intend to solve the **nonhomogeneous linear system**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{G}(t) \tag{12.1}$$

where $\Psi(t)$ is a fundamental matrix to the system $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{P}(t)\boldsymbol{x}$.

Theorem 12.2 The general solution for system(12.1) is given by:

$$\boldsymbol{x} = \Psi(t)\boldsymbol{c} + \boldsymbol{x}_{\text{particular}}$$

where

$$\boldsymbol{x}_{\text{particular}} = \Psi(t)\int \Psi^{-1}(t)\boldsymbol{G}(t)\,\mathrm{d}t$$

and $\boldsymbol{c}$ is a coefficient column vector.

Here we could use variation of parameters to show the formula for the particular solution of system(12.1):

Proof. We assume the particular solution of system(12.1) to be

$$\boldsymbol{x} = \boldsymbol{\Psi}(t)\boldsymbol{u}(t) \tag{12.2}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{\Psi}'(t)\boldsymbol{u}(t) + \boldsymbol{\Psi}(t)\boldsymbol{u}'(t) \tag{12.3}$$

$$= \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{G}(t) = \boldsymbol{P}(t)\boldsymbol{\Psi}(t)\boldsymbol{u}(t) + \boldsymbol{G}(t) \tag{12.4}$$

Since $\boldsymbol{\Psi}(t)$ is a fundamental matrix, $\boldsymbol{\Psi}'(t) = \boldsymbol{P}(t)\boldsymbol{\Psi}(t)$. We plug it into Eq.(12.3) to derive:

$$\boldsymbol{\Psi}(t)\boldsymbol{u}'(t) = \boldsymbol{G}(t)$$

Since $\boldsymbol{\Psi}(t)$ is invertible, we obtain:

$$\boldsymbol{u}'(t) = \boldsymbol{\Psi}^{-1}(t)\boldsymbol{G}(t) \implies \boldsymbol{u}(t) = \int \boldsymbol{\Psi}^{-1}(t)\boldsymbol{G}(t)\,\mathrm{d}t.$$

Hence the particular solution of system(12.1) is:

$$\boldsymbol{x} = \boldsymbol{\Psi}(t)\int \boldsymbol{\Psi}^{-1}(t)\boldsymbol{G}(t)\,\mathrm{d}t.$$

■

■ **Example 12.6** Solve the system

$$\boldsymbol{x}' = \begin{pmatrix} 4 & 8 \\ -2 & -4 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \qquad t > 0.$$

The eigenvalues of $\boldsymbol{A}$ must satisfy:

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 4-r & 8 \\ -2 & -4-r \end{vmatrix} = r^2 = 0 \implies r_1 = r_2 = 0.$$

And we find that

$$A^2 = 0 \implies \phi(t) = e^{At} = I + At = \begin{pmatrix} 4t+1 & 8t \\ -2t & -4t+1 \end{pmatrix}$$

Thus the solution to the homogeneous part is

$$x_c = c_1 \begin{pmatrix} 4t+1 \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} 8t \\ -4t+1 \end{pmatrix}.$$

It follows that

$$\phi^{-1}(t) = \begin{pmatrix} -4t+1 & -8t \\ 2t & 4t+1 \end{pmatrix}$$

$$\phi^{-1}(t)G(t) = \begin{pmatrix} -4t+1 & -8t \\ 2t & 4t+1 \end{pmatrix} \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix} = \begin{pmatrix} 8t^{-1} - 4t^{-2} + t^{-3} \\ -4t^{-1} + t^{-2} \end{pmatrix}.$$

$$\int \phi^{-1}(t)G(t)\,\mathrm{d}t = \begin{pmatrix} 8\ln t + 4t^{-1} - \frac{1}{2}t^{-2} \\ -4\ln t - t^{-1} \end{pmatrix}$$

The particular solution is given by:

$$x_{\text{particular}} = \begin{pmatrix} 4t+1 & 8t \\ -2t & -4t+1 \end{pmatrix} \begin{pmatrix} 8\ln t + 4t^{-1} - \frac{1}{2}t^{-2} \\ -4\ln t - t^{-1} \end{pmatrix} = \begin{pmatrix} 8 + 8\ln t + 2t^{-1} - \frac{1}{2}t^{-2} \\ -4 - 4\ln t \end{pmatrix}$$

The general solution is given by:

$$x = c_1 \begin{pmatrix} 4t+1 \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} 8t \\ -4t+1 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \end{pmatrix} \ln t + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-1} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-2}.$$

■

12.1.5 Method of undetermined coefficients

We could also use **undetermined coefficients method** to solve a nonhomogeneous system if

- The coefficient matrix P is constant
- The components of $G(t)$ are **polynomial**, **exponential**, or **sinusodial** functions, or **products** and **sum** of these functions.

There is a main difference between system and single equation when applying this method:
If $G(t)$ has the form ue^{rt}, where r is the eigenvalue of the coefficient matrix, we should assume the solution to be $ate^{rt} + be^{rt}$ instead of ate^{rt}.

■ **Example 12.7** Find the general solution to the system

$$x' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix} \tag{12.5}$$

The eigenvalues of A must satisfy

$$\det(A - rI) = \begin{vmatrix} 2-r & 3 \\ -1 & -2-r \end{vmatrix} = r^2 - 1 = 0 \implies r_1 = 1,\ r_2 = -1.$$

It is easy to verify the eigen-pairs are $(r_1, \boldsymbol{\xi}^{(1)})$ and $(r_2, \boldsymbol{\xi}^{(2)})$, where

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus the solution to the homogeneous part is:

$$\boldsymbol{x}_c = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.$$

The nonhomogeneous term $\boldsymbol{G}(t)$ has the form:

$$\boldsymbol{G}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$$

Hence we guess the particular solution to be:

$$\boldsymbol{x}_0 = \boldsymbol{a}te^t + \boldsymbol{b}e^t + \boldsymbol{c}t + \boldsymbol{d}.$$

It follows that

$$\boldsymbol{x}_0' = \boldsymbol{a}te^t + (\boldsymbol{a}+\boldsymbol{b})e^t + \boldsymbol{c}.$$

Substituting $\boldsymbol{x}'$ and $\boldsymbol{x}$ in Eq.(12.5), we obtain:

$$\boldsymbol{a}te^t + (\boldsymbol{a}+\boldsymbol{b})e^t + \boldsymbol{c} = \boldsymbol{A}\boldsymbol{a}te^t + \left(\boldsymbol{A}\boldsymbol{b} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^t + \left(\boldsymbol{A}\boldsymbol{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) t + \boldsymbol{A}\boldsymbol{d}.$$

It follows that

$$\begin{cases} \boldsymbol{A}\boldsymbol{a} = \boldsymbol{a} \\ \boldsymbol{A}\boldsymbol{b} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boldsymbol{a} + \boldsymbol{b} \\ \boldsymbol{A}\boldsymbol{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{0} \\ \boldsymbol{A}\boldsymbol{d} = \boldsymbol{c} \end{cases}$$

We can see that $\boldsymbol{a}$ is the eignvector corresponding to $r = 1$. We set $\boldsymbol{a} = q_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. From the second equation, we find this system could be solved only when $q_1 = \frac{1}{2}$. Thus $\boldsymbol{a} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$. In this case, we choose $\boldsymbol{b}$ to be

$$\boldsymbol{b} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

Solving for third and fourth equation, we derive

$$\boldsymbol{c} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Hence the particular solution is given by:

$$\boldsymbol{x}_{\text{particular}} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} te^t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The general solution is given by:

$$\boldsymbol{x} = \boldsymbol{x}_c + \boldsymbol{x}_{\text{particular}} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} te^t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

■

13 — Week13

13.1 Linear System and the Phase Plane

Consider equation

$$\boldsymbol{X}' = f(\boldsymbol{X}) \tag{13.1}$$

, the critical point satisfies

$$f(\boldsymbol{X}_0) = 0,$$

$\boldsymbol{X}_0$ is called critical points or equilibrium solution of Eq.(13.1).

Definition 13.1 — Stability. A critical point $\boldsymbol{X}_0$ of Eq.(13.1) is called **stable** if for $\forall \varepsilon > 0$, there $\exists \delta > 0$ such that for any solution

$$\boldsymbol{X} = \phi(t) \text{ satisfying } ||\phi(0) - \boldsymbol{X}_0|| < \delta.$$

we have $||\phi(t) - \boldsymbol{X}_0|| < \varepsilon$ for $\forall t > 0$. ■

Definition 13.2 — Asymptotically stable. A critical point $\boldsymbol{X}_0$ is called **asymptotically stable** if it is **stable** and for solution $X = \phi(t)$, there $\exists \delta > 0$ such that $||\phi(0) - \boldsymbol{X}_0|| < \delta$, we have

$$\lim_{t \to \infty} \phi(t) = \boldsymbol{X}_0.$$

■

Consider $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$, where $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$.

- Case 1: $\boldsymbol{A}$ has two **real** and **distinct** eigenvalues of the same sign.

  $$r_1, r_2 > 0, \qquad \boldsymbol{\xi}^1 \text{ and } \boldsymbol{\xi}^2 \text{ are eigenvectors associated with } r_1 \text{ and } r_2.$$

  We have:

  $$\boldsymbol{X}(t) = c_1 \boldsymbol{\xi}^1 e^{r_1 t} + c_2 \boldsymbol{\xi}^2 e^{r_2 t}.$$

- $r_1 < r_2 < 0$. The illustration is shown in Figure(13.1)

$$\boldsymbol{X}(t) = e^{r_2 t}\left[c_1\boldsymbol{\xi}^1 e^{(r_1-r_2)t} + c_2\boldsymbol{\xi}^2\right]$$

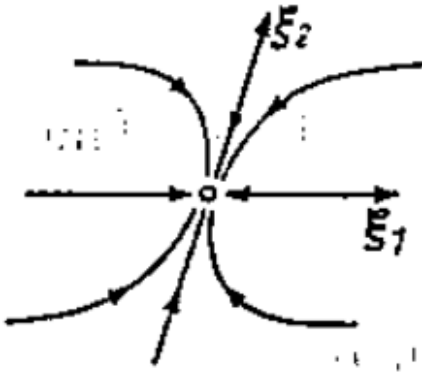


Figure 13.1: Node, $r_1 < r_2 < 0$

As $t \to \infty$, $\boldsymbol{X}(t) \to c_2\boldsymbol{\xi}^2 e^{r_2 t}$.
Direction: $\boldsymbol{\xi}^2$.
$\boldsymbol{X}(t) \to 0$ as $t \to \infty$.
Critical point: $(0,0)$ is asymptotically stable, which is called a **Node**.

- $r_1 > r_2 > 0$. The illustration is shown in Figure(13.2)

$$\boldsymbol{X}(t) = e^{r_1 t}\left[c_1\boldsymbol{\xi}^1 + c_2\boldsymbol{\xi}^2 e^{(r_2-r_1)t}\right]$$

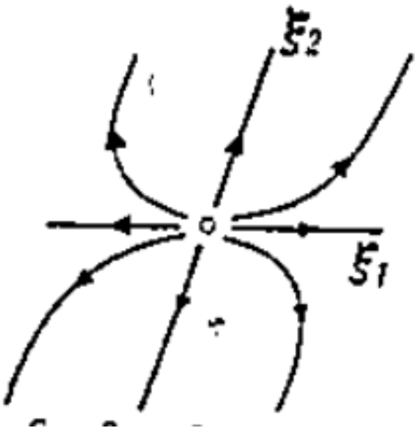


Figure 13.2: Node, $r_1 > r_2 > 0$

As $t \to \infty$, $\boldsymbol{X}(t) \to c_1\boldsymbol{\xi}^1 e^{r_1 t}$.
Direction: $\boldsymbol{\xi}^1$
$\boldsymbol{X}(t) \to \infty$ as $t \to \infty$.
Critical point: $(0,0)$ is unstable, which is called a **Node**.

- ***A*** has two real distinct eigenvalues of the opposite sign, $r_1 > 0 > r_2$. The illustration is shown in Figure(13.3)

$$\boldsymbol{X}(t) = e^{r_1 t}\left[c_1\boldsymbol{\xi}^1 + c_2\boldsymbol{\xi}^2 e^{(r_2-r_1)t}\right]$$

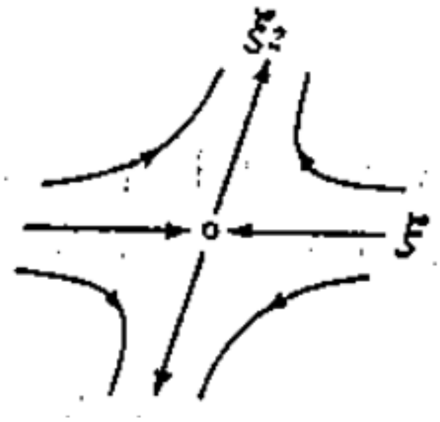


Figure 13.3: saddle point, $r_1 > 0 > r_2$

As $t \to \infty$, $\boldsymbol{X}(t) \to c_1\boldsymbol{\xi}^1 e^{r_1 t}$.
Direction: $\boldsymbol{\xi}^1$, $\boldsymbol{X}(t) \to \infty$.
Critical point: $(0,0)$ is unstable, which is called a **saddle point**.

- ***A*** hsa two repeated eigenvalues $r_1 = r_2 = r$.
  - two lienarly independent eigenvectors.

$$\boldsymbol{X}(t) = e^{rt}\left(c_1\boldsymbol{\xi}^1 + c_2\boldsymbol{\xi}^2\right)$$

Each trojectory in the phase plane must be a straight line.
  * when $r > 0$, $(0,0)$ is unstable. The illustration is shown in Figure(13.4)

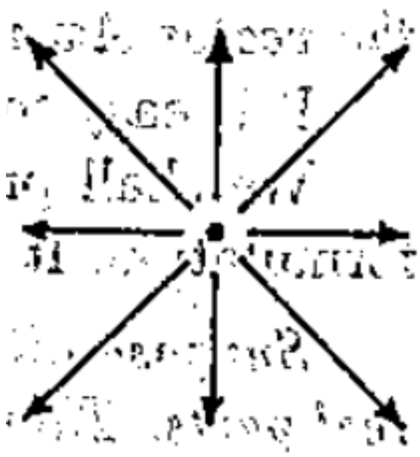


Figure 13.4: Unstable, $r_1 = r_2 = r > 0$ (two linearly independent eigenvectors)

∗ when $r < 0$, $(0,0)$ is AS, which is called a **proper node**. The illustration is shown in Figure(13.5)

Figure 13.5: Proper node, $r_1 = r_2 = r < 0$ (two linearly independent eigenvectors)

– One linearly independent eigenvector $\boldsymbol{\xi}$ associated with r.

$$\boldsymbol{X}(t) = c_1\boldsymbol{\xi}e^{rt} + c_2\left[\boldsymbol{\xi}t + \boldsymbol{\eta}\right]e^{rt} = e^{rt}\left[c_1\boldsymbol{\xi} + c_2\left(\boldsymbol{\xi}t + \boldsymbol{\eta}\right)\right]$$

As $t \to \infty$, $\boldsymbol{X}(t) \to e^{rt}C_2\boldsymbol{\xi}t$.

Direction: $\boldsymbol{\xi}$

∗ $r > 0$, $\boldsymbol{X}(t \to \infty)$,$(0,0)$ is unstable. The illustration is shown in Figure(13.6)

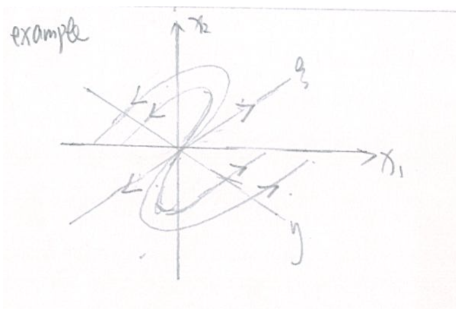


Figure 13.6: Unstable, $r_1 = r_2 = r > 0$, (only one linearly independent eigenvector)

∗ $r < 0$, $\boldsymbol{X}$ is stable, which is called a **improper node**. The illustration is shown in Figure(13.7)

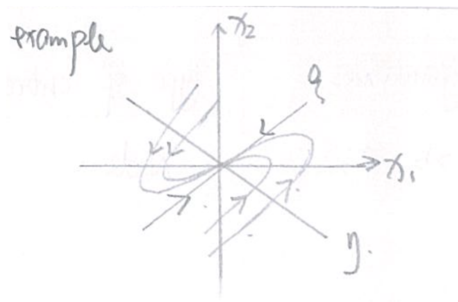


Figure 13.7: Node, $r_1 = r_2 = r < 0$, (only one linearly independent eigenvector)

■ **Example 13.1** $\boldsymbol{X}' = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$, we get

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = (r+2)^2.$$

Thus $r_1 = r_2 = -2$, $\boldsymbol{\xi} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\boldsymbol{\eta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

$$\boldsymbol{X}(t) = e^{-2t}\left\{c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]\right\}$$

■

- Case 4: $\boldsymbol{A}$ has two complex eigenvalues, $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\alpha \neq 0$.

$$\boldsymbol{\xi}^1 = \boldsymbol{u} + i\boldsymbol{v}, \quad \boldsymbol{\xi}^2 = \boldsymbol{u} - i\boldsymbol{v}.$$

$\boldsymbol{X}(t) = c_1\Re(\boldsymbol{\xi}^1 e^{r_1 t}) + c_2\Im(\boldsymbol{\xi}^1 e^{r_1 t})$ Notice that

$$\begin{aligned} \boldsymbol{\xi}^1 e^{r_1 t} &= (\boldsymbol{u} + i\boldsymbol{v})e^{\alpha t}(\cos(\beta t) + \sin(\beta t)) \\ &= e^{\alpha t}\left[(\boldsymbol{u}\cos(\beta t) - \boldsymbol{v}\sin(\beta t)) + i(\boldsymbol{u}\sin(\beta t) + \boldsymbol{v}\cos(\beta t))\right] \end{aligned}$$

Thus

$$\boldsymbol{X}(t) = c_1\left[\boldsymbol{u}\cos(\beta t) - \boldsymbol{v}\sin(\beta t)\right]e^{\alpha t} + c_2\left[\boldsymbol{u}\sin(\beta t) + \boldsymbol{v}\cos(\beta t)\right]e^{\alpha t}$$

$\boldsymbol{X}(t)$ rotates. When $\beta > 0$, from $\boldsymbol{u}$ to $-\boldsymbol{v}$, clockwise. When $\beta < 0$, from $\boldsymbol{u}$ to $\boldsymbol{v}$, anticlockwise.
When $\alpha > 0$, the $\boldsymbol{X}(t)$ expands, when $\alpha < 0$, the $\boldsymbol{X}(t)$ shrinks.

– When $\alpha > 0, \beta > 0$, unstable. The illustration is shown in Figure(13.8).

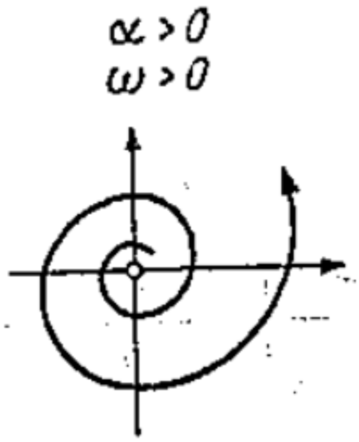


Figure 13.8: Unstable, $\alpha > 0, \beta > 0$

– When $\alpha < 0, \beta > 0$, AS. The illustration is shown in Figure(13.9).

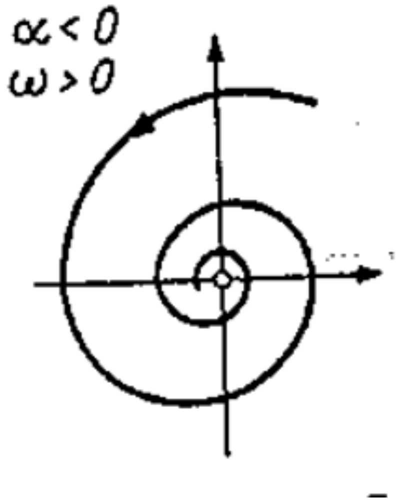


Figure 13.9: AS, $\alpha < 0, \beta > 0$

– When $\alpha > 0, \beta < 0$, unstable. The illustration is shown in Figure(13.10).

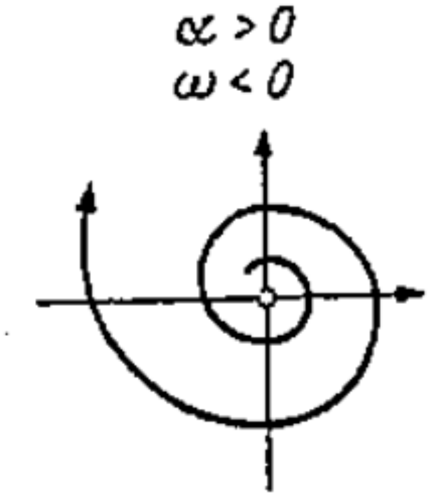


Figure 13.10: unstable, $\alpha > 0, \beta < 0$

– When $\alpha < 0, \beta < 0$, AS. The illustration is shown in Figure(13.11).

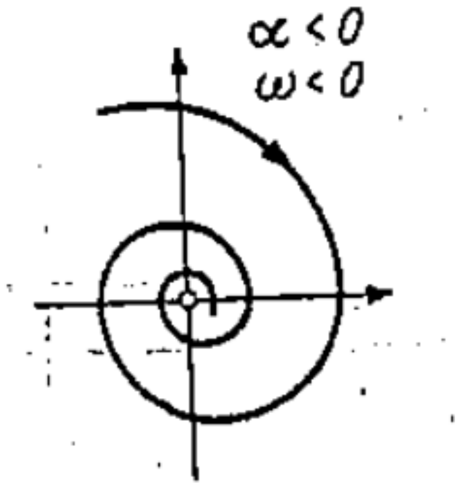


Figure 13.11: AS, $\alpha < 0, \beta < 0$

These points are all called **sprial point**.

- Case 5: $\boldsymbol{A}$ has two complex eigenvalues with pure imaginary parts. $r_1 = i\beta, r_2 = -i\beta$.

$$\boldsymbol{X}(t) = c_1\left[\boldsymbol{u}\cos(\beta t) - \boldsymbol{v}\sin(\beta t)\right] + c_2\left[\boldsymbol{u}\sin(\beta t) + \boldsymbol{v}\cos(\beta t)\right]$$

stable but not AS!

  - when $\beta > 0$, the illustration is shown in Figure(13.12).

Figure 13.12: Center, $\beta > 0$

  - When $\beta < 0$, the illustration is shown in Figure(13.13).

Figure 13.13: Center, $\beta < 0$

$(0,0)$ is a center!

R Summary:

- When $Re(r_1) < 0$ and $Re(r_2) < 0$, AS!
- When $Re(r_1) > 0$ or $Re(r_2) > 0$, unstable!
- Purely imaginary, stable but not AS!

13.2 Locally linear system

$$x' = f(x) \tag{13.2}$$

suppose x_0 is a critical point, i.e.

$$f(x_0) = \mathbf{0}.$$

System(13.2) is called **locally linerar** near critical point x_0 if

$$\begin{cases} f(x) = A(x - x_0) + g(x), \qquad \det(A) \neq 0 \\ \lim\limits_{x \to x_0} \dfrac{||g(x)||}{||x - x_0||} = 0. \end{cases}$$

In particular, if $x_0 = \mathbf{0}$, then system(13.2) is **locally linear** near $\mathbf{0}$ if

$$f(x) = Ax + g(x), \qquad \det(A) \neq 0.$$

$$\lim_{x \to x_0} \frac{||g(x)||}{||x||} = 0.$$

The linear system of (13.2) is

$$x' = A(x - x_0)$$

near critical point x_0.
If f is continuously differentiable near x_0, then $A = \triangledown f(x)$.

■ **Example 13.2** Consider the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -w^2 \sin x - \gamma y \end{cases} \quad (r > 0, w > 0)$$

Solution. • Firstly, we find the critical point:

$$\begin{cases} y = 0 \\ -w^2 \sin x - \gamma y = 0. \end{cases} \Longrightarrow \begin{cases} x = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots \\ y = 0 \end{cases}$$

It follows that

$$f(x,y) = \begin{pmatrix} y \\ -w^2 \sin x - \gamma y \end{pmatrix} \Longrightarrow \triangledown f(x,y) = \begin{pmatrix} 0 & 1 \\ -w^2 \cos x & -\gamma \end{pmatrix}$$

Hence

$$\triangledown f(k\pi, y) = \begin{pmatrix} 0 & 1 \\ (-1)^{k+1} w^2 & -\gamma \end{pmatrix}$$

The linearized system near $(k\pi, 0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ (-1)^{k+1} w^2 & -\gamma \end{pmatrix} \begin{pmatrix} x - k\pi \\ y \end{pmatrix}$$

- For $k = 0$, critical point is $(0,0)$.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -w^2 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are computed as:

$$0 = \begin{vmatrix} -\lambda & 1 \\ -w^2 & -\gamma - \lambda \end{vmatrix} = \lambda^2 + \gamma\lambda + w^2$$

If $\gamma^2 - 4w^2 > 0$,

$$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4w^2}}{2}, \lambda_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4w^2}}{2}$$

If $\gamma^2 - 4w^2 < 0$,

$$\lambda_1 = \frac{-\gamma + i\sqrt{4w^2 - \gamma^2}}{2}, \lambda_2 = \frac{-\gamma - i\sqrt{4w^2 - \gamma^2}}{2}$$

If $\gamma^2 - 4w^2 = 0$,

$$\lambda_1 = \lambda_2 = -\frac{\gamma}{2}.$$

In conclusion, $(0,0)$ is AS!

- For $k = 1$, critial point is $(\pi, 0)$.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ w^2 & -\gamma \end{pmatrix} \begin{pmatrix} x - \pi \\ y \end{pmatrix}$$

It follows that

$$0 = \begin{vmatrix} -\lambda & 1 \\ w^2 - \gamma & -\lambda \end{vmatrix} = \lambda^2 + \gamma\lambda - w^2$$

If $\gamma^2 + 4w^2 > 0$,

$$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2 + 4w^2}}{2}, \lambda_2 = \frac{-\gamma - \sqrt{\gamma^2 + 4w^2}}{2}$$

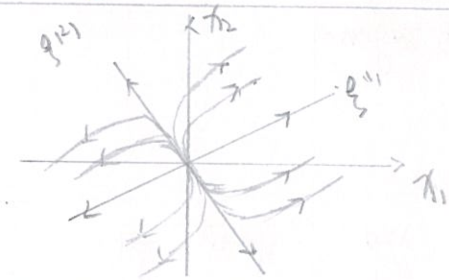
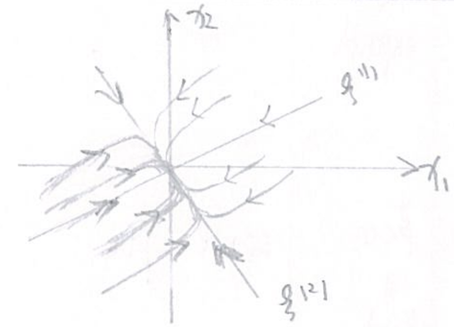
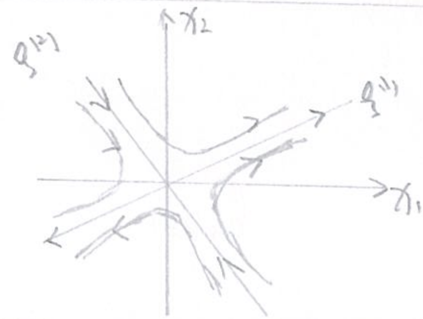
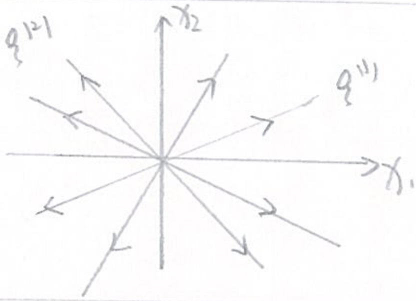
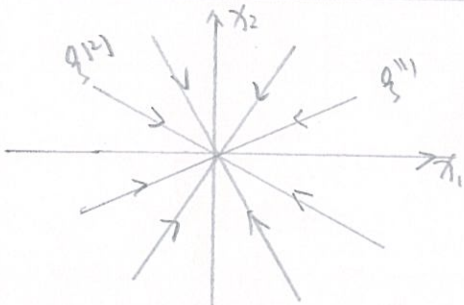
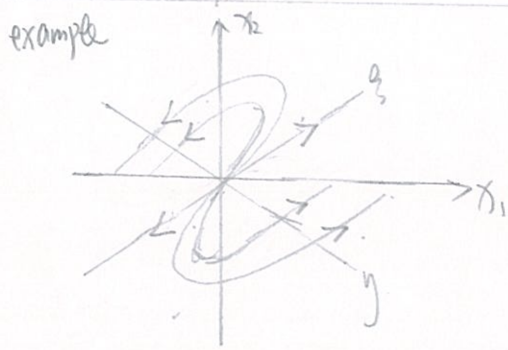
In conclusion, $(\pi, 0)$ is unstable and...

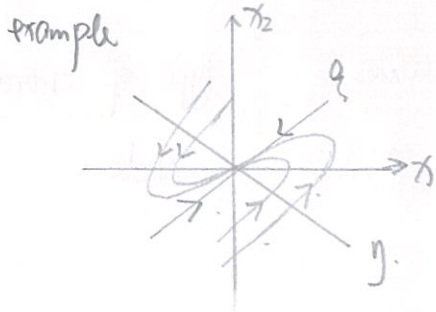
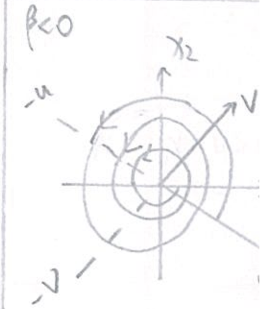
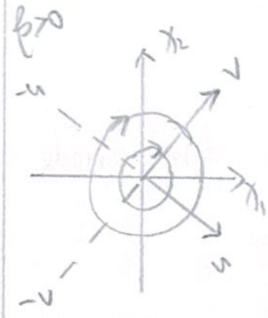
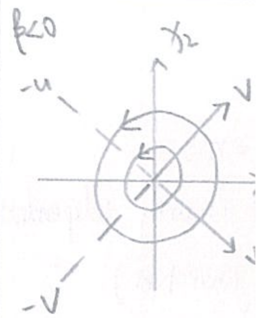
If k is an even integer, then $(k\pi, 0)$ are AS. If k is an odd integer, then $(k\pi, 0)$ are unstable.

■

R Conclusion:

1. If eigenvalues are not purely imaginary, then stability & type of the original locally linear system = stability & type of linearized system.
2. If the critial points of lineraized system is a center, then stability & type of original locally linear system cannot be drawn from its lienarized system.

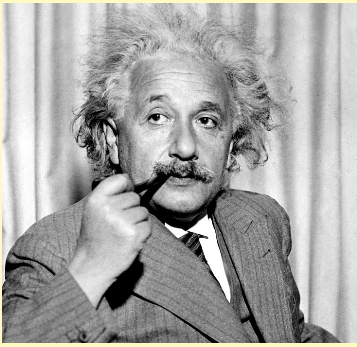
eigenvalues	type of critical pt	stability	trajectories.
$r_1 > r_2 > 0$	Node	unstable	
$r_1 < r_2 < 0$	Node	Asymptotically stable	
$r_2 < 0 < r_1$	saddle point	unstable	
$r_1 = r_2 > 0$ (2 linearly independent eigenvectors).	proper node	unstable	
$r_1 = r_2 < 0$ (2 linear independent eigenvectors)	proper node	AS.	
$r_1 = r_2 > 0$ (only 1 linearly independent eigen-vector)	improper node	unstable	

$r_1 = r_2 < 0$ (only 1 linearly independent eigen-vector).	improper node	A.S.	example 
$r = \lambda \pm i\beta$, $\lambda > 0$. $\Downarrow$ expand	spiral pt.	unstable	 
$r = \lambda \pm i\beta$, $\lambda < 0$ $\Downarrow$ shrink.	spiral pt	A.S.	 
$r = \pm i\beta$	center	stable	 

Ordinary Differential Equation

MAT2002 Notebook

Grade Descriptor of A – Outstanding performance on solving linear or nonlinear ordinary differential equations and has the ability to synthesize and apply the basic theory of ODEs to novel situations in a manner that would surpass the expectation at this level or could be common at higher levels of study or research.
Grade Descriptor of F – Unsatisfactory performance on a number of learning outcomes, or failure to meet specified assessment requirements. In other words, you perform too bad on final!



Walter Rudin is a person who writes this book using LaTeX. He is interested in Mathematics. Recently he is working on Information Theory and Graph Theory. You can contact with him on these fields. But he is very carelessness. If you find some typos in this book, don't hesitate to ask him directly. Hope you enjoy the journey to Math!

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