ISyE 3770 Assignment 5: Point Estimator

Due date: 11:59 PM, Friday, March 29, 2024.

Question 1 (MSE). *Let X_1, X_2 be independent random variables with mean μ and variance σ^2. Suppose we have two estimators of μ:*

$$\widehat{\Theta}_1 = \frac{X_1 + X_2}{2}$$
$$\widehat{\Theta}_2 = \frac{X_1 + 3X_2}{4}.$$

1) *Are both estimators unbiased?* *(8 points)*
2) *What is the variance of each estimator?* *(8 points)*
3) *What is the MSE of two estimators?* *(8 points)*

Solution.. 1) Yes. It can be verified that $\mathbb{E}[\widehat{\Theta}_1] = \mu$ and $\mathbb{E}[\widehat{\Theta}_2] = \mu$.

2) You can check that

$$\text{Var}(\widehat{\Theta}_1) = \frac{1}{4}\text{Var}(X_1) + \frac{1}{4}\text{Var}(X_2) = \frac{1}{2}\sigma^2$$

and

$$\text{Var}(\widehat{\Theta}_2) = \frac{1}{16}\text{Var}(X_1) + \frac{9}{16}\text{Var}(X_2) = \frac{5}{8}\sigma^2.$$

3) Recall that $\text{MSE} = (\text{Bias})^2 + \text{Variance}$. Since these two estimators are unbiased, it holds that

$$\text{MSE}(\widehat{\Theta}_1) = \text{Var}(\widehat{\Theta}_1) = \frac{\sigma^2}{2}, \quad \text{MSE}(\widehat{\Theta}_2) = \text{Var}(\widehat{\Theta}_2) = \frac{5\sigma^2}{8}.$$

□

Question 2 (MSE). *Suppose $X \sim$ Uni$(\theta, 3\theta)$ with $\theta > 0$. Let $X_1, \ldots, X_n$ be n i.i.d. random variables following the same distribution as X.*

1) *Prove that $\frac{\overline{X}}{2}$ is an unbiased estimator of θ.* *(8 points)*
2) *Calculate the MSE of $\frac{\overline{X}}{2}$ and $\overline{X}$.* *(8 points)*

Solution. 1) You can check that $\mathbb{E}[\frac{\overline{X}}{2}] = \frac{1}{2}\mathbb{E}[X] = \theta$.

2) Recall that $\text{MSE} = (\text{Bias})^2 + \text{Variance}$. Therefore,

$$\text{MSE}(\frac{\overline{X}}{2}) = \text{Var}(\frac{\overline{X}}{2}) = \frac{1}{4} \cdot \frac{\text{Var}(X)}{n} = \frac{1}{4}\frac{4\theta^2}{12n} = \frac{\theta^2}{12n}.$$

Moreover, $\text{Bias}(\overline{X}) = \theta$ and therefore,

$$\text{MSE}(\overline{X}) = \theta^2 + \text{Var}(\overline{X}) = \theta^2 + \frac{\theta^2}{3n} = (1 + \frac{1}{3n})\theta^2.$$

□

Question 3 (Maximum Likelihood Estimator)**.** *A random variable X has the following probability density function:*

$$f(x;\theta) = \frac{1}{2\theta^3}x^2 e^{-x/\theta}, \qquad 0 < x < \infty, 0 < \theta < \infty.$$

Let $X_1, \ldots, X_n$ be n i.i.d. random variables following the same distribution as X. Find the maximum likelihood estimator for θ. *(15 points)*

Solution.

- Step 1: write down the log-likelihood function and simplify it:

$$\begin{aligned} l(\theta; X_1, \ldots, X_n) &= \sum_{i=1}^n \log f(X_i;\theta) = \sum_{i=1}^n \log\left[\frac{1}{2\theta^3}X_i^2 e^{-X_i/\theta}\right] \\ &= \sum_{i=1}^n \left[-3\log\theta - \frac{X_i}{\theta}\right] + \text{Constant} \\ &= -3n\log\theta - \frac{\sum_{i=1}^n X_i}{\theta} + \text{Constant.} \end{aligned}$$

- Step 2: find the maximizer of $l(\theta; X_1, \ldots, X_n)$: let $\widehat{\theta}$ be the maximizer of $l(\theta; X_1, \ldots, X_n)$, then

$$\frac{\partial l(\widehat{\theta}; X_1, \ldots, X_n)}{\partial\theta} = \frac{-3n}{\widehat{\theta}} + \frac{\sum_{i=1}^n X_i}{\widehat{\theta}^2} = 0 \implies \widehat{\theta} = \frac{\overline{X}}{3}.$$

□

Question 4 (Maximum Likelihood Estimator)**.** *A random variable X has the following probability density function:*

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Let $X_1, \ldots, X_n$ be n i.i.d. random variables following the same distribution as X. Find the maximum likelihood estimator for μ and σ^2. *(15 points)*

Solution.

- Step 1: write down the log-likelihood function and simplify it:

$$\begin{aligned} l(\mu,\sigma^2; X_1, \ldots, X_n) &= \sum_{i=1}^n \log f(X_i;\mu,\sigma^2) = \sum_{i=1}^n \log\left[\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right] \\ &= \sum_{i=1}^n \left[-\frac{1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(X_i-\mu)^2\right] + \text{Constant} \\ &= -\frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (X_i-\mu)^2 + \text{Constant.} \end{aligned}$$

- Step 2: find the maximizer of $l(\mu,\sigma^2; X_1, \ldots, X_n)$: let $\widehat{\mu}, \widehat{\sigma}^2$ be the maximizer of $l(\mu,\sigma^2; X_1, \ldots, X_n)$, then

$$\frac{\partial l(\widehat{\mu},\widehat{\sigma}^2; X_1, \ldots, X_n)}{\partial\mu} = \frac{1}{\widehat{\sigma}^2}\sum_{i=1}^n (X_i - \widehat{\mu}) = 0 \implies \widehat{\mu} = \overline{X},$$

  and

$$\frac{\partial l(\widehat{\mu},\widehat{\sigma}^2; X_1, \ldots, X_n)}{\partial\sigma^2} = -\frac{n}{2\widehat{\sigma}^2} + \frac{1}{2\widehat{\sigma}^4}\sum_{i=1}^n (X_i-\widehat{\mu})^2 = 0 \implies \widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \widehat{\mu})^2.$$

□

Question 5 (Maximum Likelihood Estimator and Method of Moment Estimator)**.** *Let X be an exponential random variable with parameter λ. Let $X_1, \ldots, X_n$ be n i.i.d. random variables following the same distribution as X.*

1) *Find the maximum likelihood estimator for* λ. *(15 points)*

2) *Find the method of moment estimator for* λ. *(15 points)*

Solution. 1) • Step 1: write down the log-likelihood function and simplify it:

$$
\begin{aligned}
l(\lambda; X_1, \ldots, X_n) &= \sum_{i=1}^{n} \log f(X_i; \lambda) = \sum_{i=1}^{n} \log \left[\lambda e^{-\lambda X_i}\right] \\
&= \sum_{i=1}^{n} [\log \lambda - \lambda X_i] + \text{Constant} \\
&= n \log \lambda - \lambda \sum_{i=1}^{n} X_i + \text{Constant.}
\end{aligned}
$$

• Step 2: find the maximizer of $l(\lambda; X_1, \ldots, X_n)$: let $\widehat{\lambda}$ be the maximizer of $l(\lambda; X_1, \ldots, X_n)$, then

$$
\frac{\partial l(\widehat{\lambda}; X_1, \ldots, X_n)}{\partial \lambda} = \frac{n}{\widehat{\lambda}} - \sum_{i=1}^{n} X_i = 0 \implies \widehat{\lambda} = \frac{1}{\overline{X}}.
$$

2) Based on the method of moment, we find the sample mean equals $\overline{X}$, and the population mean equals $\frac{1}{\lambda}$. Therefore,

$$
\overline{X} = \frac{1}{\widetilde{\lambda}} \implies \widetilde{\lambda} = \frac{1}{\overline{X}}.
$$

□