ISyE 3770, Spring 2024
Statistics and Applications

Introduction to Continuous Distribution (I)

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- Chapter 3 Continuous Distribution

  - Section 3.1 RV of the continuous type

Motivation: RVs with continuous range of possible values are common in practice.

Definition [Continuous RV]

A RV $X: S \to X(S)$ is said to be continuous if there exists a function $f(x): X(S) \to [0, +\infty)$ such that

1. $f(x) \geq 0, x \in X(S)$
2. $\int_{X(S)} f(x) dx = 1$
3. If $(a, b) \subseteq X(S)$, then $P(a \leq X \leq b) = \int_a^b f(x)\, dx$.

Here, $f(x)$ is called the probability density function (pdf) of X.

Remark:

- We often extend the domain of $f(x)$ from X(S) to R and let $f(x)=0$ for $x \notin$ X(S). From now on, we consider pdf $f(x)$: R $\longrightarrow [0,+\infty)$ .X(S) is called the **support** of $f(x)$.
- Then the 3 conditions become:

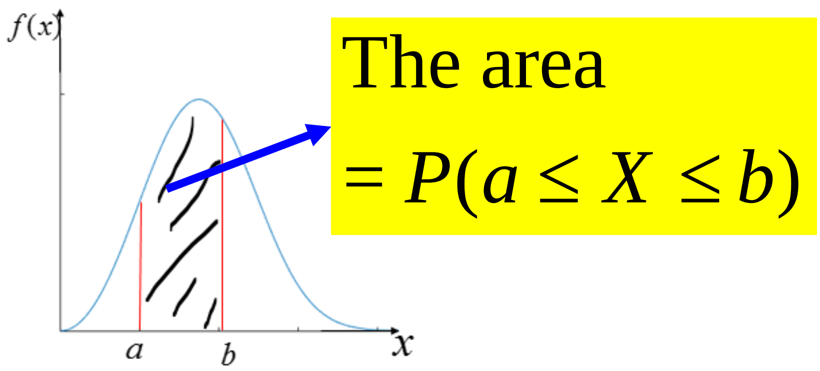
➢ $f(x) \geq 0$ for $x \in R$

➢ $\int_{-\infty}^{+\infty} f(x)dx = 1$

➢ $P(a \leq X \leq b) = \int_a^b f(x)\, dx.$

- For any single value a, $P(X = a) = \int_a^a f(x)dx = 0.$ Therefore, including or excluding the endpoints of an interval has **no** effect on its probability:

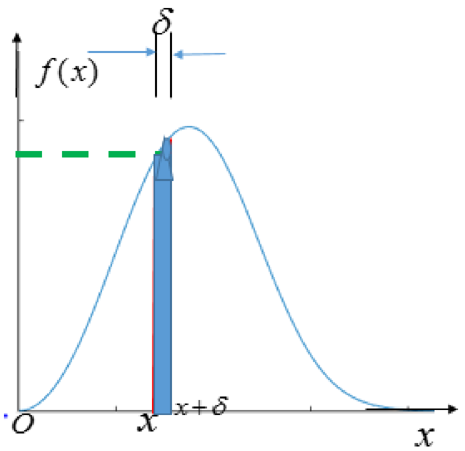
$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b).$



- Interpretation of pdf

For very small $\delta > 0$,

$$P([x, x+\delta]) = \int_x^{x+\delta} f(x)dx \approx f(x)\delta.$$



The *pdf* *f*(*x*) in the picture can be seen as the probability mass per unit length near *x*.

Remark:

$$\frac{d[F(x)]}{dx} = f(x).$$

Definition [Cumulative distribution function (cdf)]

The *cumulative distribution function* or ***cdf*** of a **continuous** RV *X*, denoted by *F*(*x*), is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

F(*x*) accumulates (or, more simply, cumulates) all of the probability less than or equal to *x*.

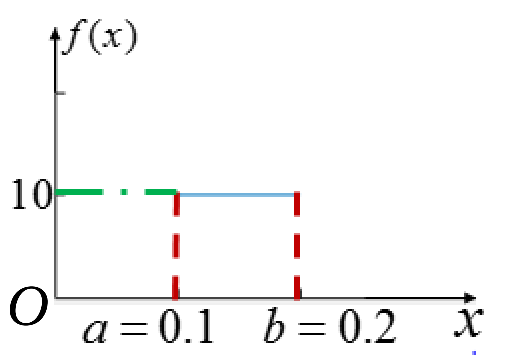
Example 1 [Uniform distribution]:

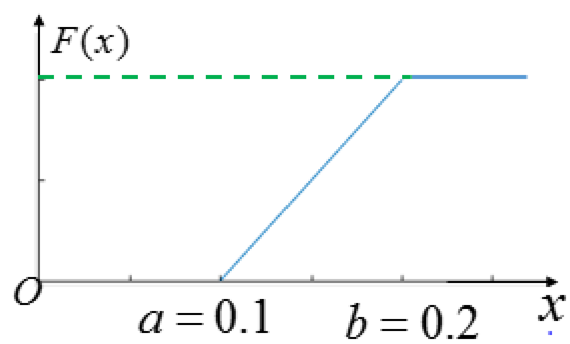
Let the random variable X denote the outcome when a point is selected randomly from $[a, b]$ with $-\infty < a < b < \infty$.

Define the *pdf* of X:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b. \\ 0, & otherwise. \end{cases} \Rightarrow F(x) = \begin{cases} 0, & x < a. \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & x > b. \end{cases}$$

$P(X \le x) = \frac{x-a}{b-a}$ implies the probability of selecting a point from the interval $[a,x]$ is proportional to the length of the interval $[a, x]$.





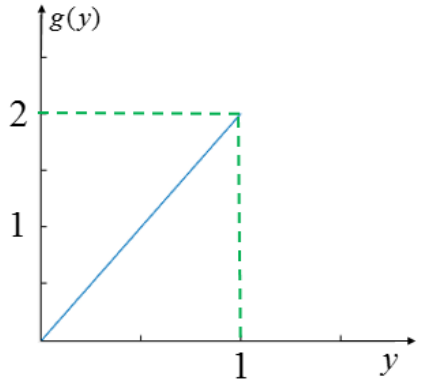
Uniform distribution: when a pmf is constant over the support.

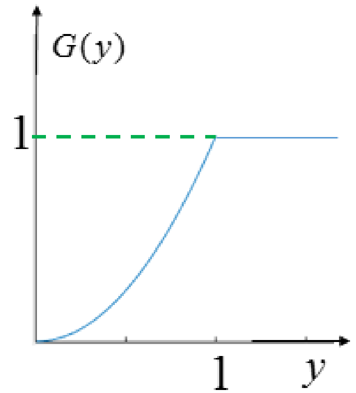
denoted by $X \sim U(a, b)$

Example 2:

Let Y be a continuous random variable with *pdf* $g(y) = 2y$, $0 < y < 1$. Then the *cdf* of Y is:

$$F(y) = P(Y \le y) = \int_{-\infty}^{y} g(t)dt = \begin{cases} 0, & y \le 0 \\ y^2, & 0 < y < 1 \\ 1, & y \ge 1 \end{cases}$$





computations of probabilities:

$$P(\frac{1}{2} < Y \le \frac{3}{4}) = F(\frac{3}{4}) - F(\frac{1}{2}) = \frac{9}{16} - \frac{1}{4} = \frac{5}{16}.$$

$$P(\frac{1}{4} \le Y < 2) = F(2) - F(\frac{1}{4}) = 1 - \frac{1}{16} = \frac{15}{16}.$$

➢ Mathematical expectation

Definition [Expectation]

Assume X is a **continuous** RV with range space X(S) and $f(x)$ is its pdf. If $\int_{X(S)} g(x)f(x)dx$ exists, then it's called the **expectation** or the **expected value** of $g(X)$ and is denoted by $E[g(X)]$. That is,

$$E[g(X)] = \int_{X(S)} g(x)f(x)dx$$

Remark:

- Mathematical expectation is a linear operator. In other words,

  $If \quad c_1 \quad and \quad c_2 \quad are \quad constants, \quad g_1(x) \quad and \quad g_2(x) \quad are \quad functions,$

  $$E\left[c_1 g_1(x) + c_2 g_2(x)\right] = c_1 E\left[g_1(x)\right] + c_2 E\left[g_2(x)\right]$$

- Letting $f(x)=0$ for $x \notin X(S)$, then we find the expectation for function g(x) can be expressed as:

  $$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

For a continuous RV X with pdf $f(x)$:

- Mean of X:

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$$

- Variance of X:

$$Var(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x) dx = E[(X-\mu)^2]$$

- Standard deviation of X:

$$\sigma = \sqrt{Var(X)}$$

- Moment generating function: if it exists, then

$$M(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x) dx, \quad -h < t < h \text{ for some } h > 0.$$

It completely determines the distribution of X and all moments exist and are finite:

$$M'(0) = E(X), M''(0) = E(X^2)$$

- Moment of X:

$$E[X^r] = \int_{-\infty}^{+\infty} x^r f(x) dx$$

Example 3:

Let X have the pdf

$$f(x) = \begin{cases} \dfrac{1}{100}, & 0 < x < 100. \\ 0, & otherwise. \end{cases} \Longleftrightarrow X \sim U(0,100)$$

Compute $E(X)$ and $Var(X)$.

$$E(X) = \int_0^{+\infty} xf(x)dx$$

$$= \int_0^{100} x \cdot \frac{1}{100} dx = \frac{1}{100}\left[\frac{x^2}{2}\right]_0^{100} = 50.$$

$$Var(X) = E\left(\left[X - E(X)\right]^2\right) = \int_0^{100} (x-50)^2 \cdot \frac{1}{100} dx = \frac{2500}{3}.$$

Actually, for $X \sim U(a,b)$

$$E(X) = \frac{a+b}{2}, \qquad Var(X) = \frac{(b-a)^2}{12}, \qquad M(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Example 4:

Let X be a **continuous** RV and have the pdf

$$f(x)=\begin{cases} xe^{-x}, & 0<x<\infty. \\ 0, & otherwise. \end{cases}$$

Compute $E(X)$ and $Var(X)$.

$Solution:$

$$M(t)=E(e^{tX})=\int_{-\infty}^{+\infty} e^{tx} f(x)dx=\int_0^{\infty} xe^{-x}e^{tx}dx$$

$$=\lim_{b\to\infty}\int_0^b xe^{-(1-t)x}dx=\lim_{b\to\infty}\left[-\frac{xe^{-(1-t)x}}{1-t}-\frac{e^{-(1-t)x}}{(1-t)^2}\right]_0^b$$

$$=\lim_{b\to\infty}\left[-\frac{be^{-(1-t)b}}{1-t}-\frac{e^{-(1-t)b}}{(1-t)^2}\right]+\frac{1}{(1-t)^2}$$

$$=\frac{1}{(1-t)^2}. \qquad \leftarrow when \;\; t<1 \Leftrightarrow 1-t>0.$$

$$M'(t)=2(1-t)^{-3}\Rightarrow M'(0)=2.$$

$$M''(t)=6(1-t)^{-4}\Rightarrow M''(0)=6.$$

$$E(X)=M'(0)=2. \qquad Var(X)=E(X^2)-[E(X)]^2=M''(0)-[M'(0)]^2=2.$$

From the above examples,
We observe that $f(x)$ is not restricted to be "$f(x)\le 1$".
And actually,
$f(x)$ needn't to be continuous.
For example,

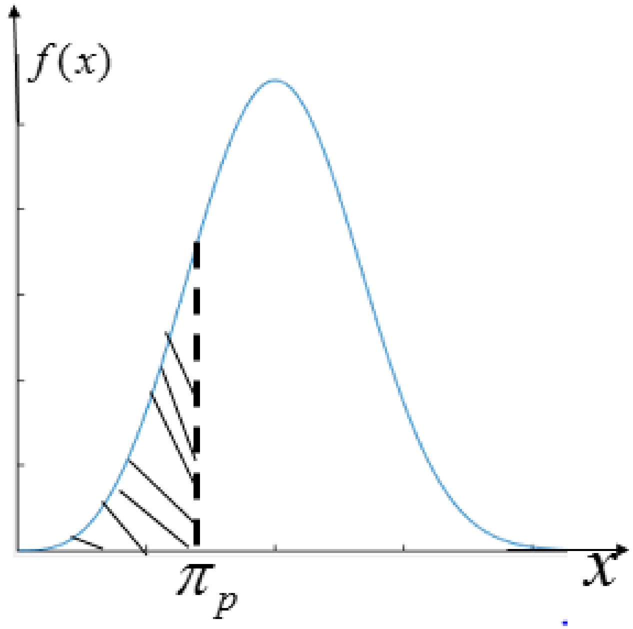
$$f(x)=\begin{cases} \frac{1}{2}, & x\in(0,1)\cup(2,3). \\ 0, & \text{otherwise.} \end{cases}$$

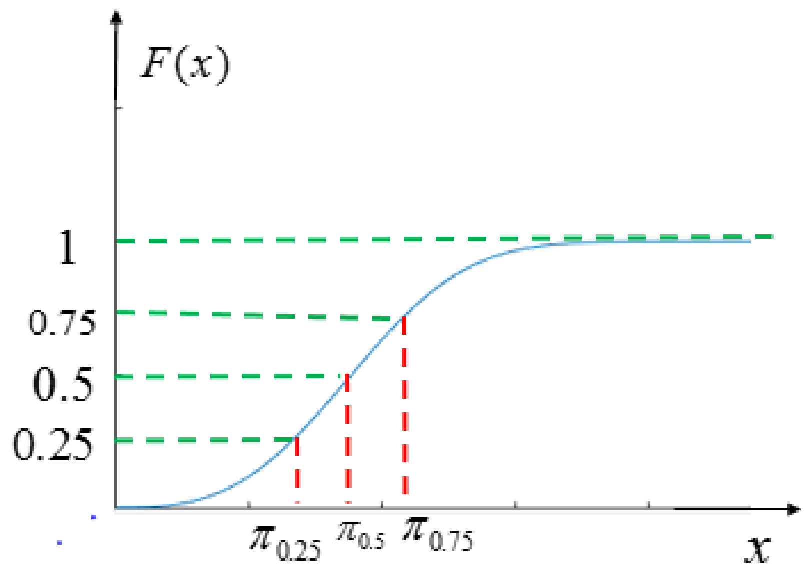
Definition 3.1-3[(100*p*)th percentile]

It is a number π_p such that the area **under *f* (*x*) to the left of $\boldsymbol{\pi_p}$** is *p*. That is,

$$p = \int_{-\infty}^{\pi_p} f(x)dx = F(\pi_p)$$

The 50th percentile is called the **median**. The 25th and 75th percentiles are called the **first and third quantiles**, respectively. The median is called the 2nd **quantile**.





Example 5:

Let X be a **continuous** RV and have the pdf

$$f(x) = \frac{3x^2}{4^3} e^{-(x/4)^3} \qquad 0 < x < +\infty$$

Compute 30th and 90th percentile.

Solution :

$$0.3 = \int_0^{\pi_{0.3}} f(x)dx$$

$$= \int_0^{\pi_{0.3}} (3x^2/4^3) e^{-(x/4)^3} dx$$

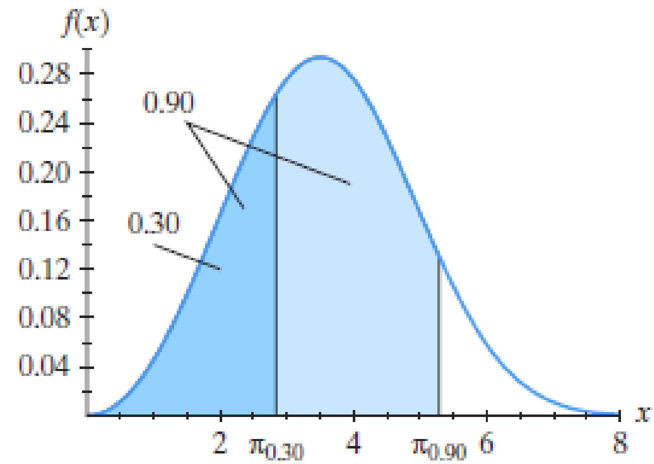
$$= \int_0^{\pi_{0.3}} e^{-(x/4)^3} d\left(x/4\right)^3$$

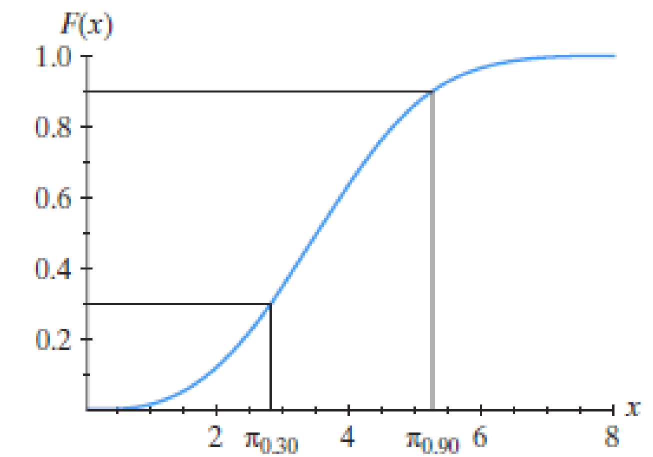
$$= \left[-e^{-u}\right]_0^{(\pi_{0.3}/4)^3}$$

$$= 1 - e^{-(\pi_{0.3}/4)^3} = 0.3$$

$$\Rightarrow \pi_{0.3} = -4(\ln 0.7)^{1/3}.$$

Similarly, $\pi_{0.9} = -4(\ln 0.1)^{1/3}$.





Chapter 3 Continuous distribution

Section 3.2 exponential, gamma, Chi-Square Distributions

- Poisson distribution.

It can be used to describe the number of occurrences of the same event in a given continuous interval with pmf $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,\dots$ $E(X) = \lambda,$ $Var(X) = \lambda.$

Now consider the *APP* with mean number of occurrences λ in a unit interval: frequency

- For an interval with length T, the number of occurrence, say X , has $E(X) = \lambda T$
- And thus its pmf is $f(x) = \frac{(\lambda T)^x e^{-\lambda T}}{x!}, x = 0,1,\dots$
- $P(X = 0) = e^{-\lambda T} = P($no occurrence in the interval with length T)

c.n.t

Let W denote the waiting time until the first occurrence during the *APP*.

- pdf of W

Idea:
① Derive *cdf* of W: $F(w)$.
② $f(w)=\frac{d[F(w)]}{dw}$

$F(w) = P(W \le w)$ Assume that the waiting time is nonnegative. Then

$F(w) = 0$ for $w < 0$.

For $w \ge 0$, $F(w) = P(W \le w) = 1 - P(W > w)$.

where $P(W > w) = P(\text{no occurrences in } [0, w]) = e^{-\lambda w}$.

Therefore, $F(w) = 1 - e^{-\lambda w}$ for $w \ge 0$

$\Rightarrow f(w) = F'(w) = \lambda e^{-\lambda w}, w \ge 0.$

What is λ?

The *mean* number of occurrences per unit interval is λ

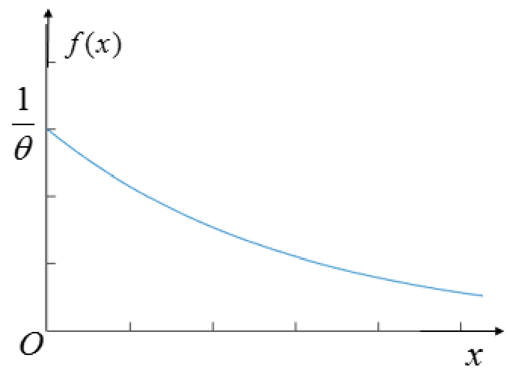
We often let $\lambda = 1/\theta$ and say that the RV has an **exponential distribution** :

Definition [Exponential distribution]

A *RV X* has an exponential distribution if its pdf is defined by

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \qquad x \geq 0, \theta > 0.$$

Accordingly, the waiting time W until the first occurrence in a Poisson process has an exponential distribution with $\theta = 1/\lambda$.



- mgf, mean and variance

$$M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = \left[\frac{1}{\theta} \frac{1}{t - 1/\theta} e^{(t-1/\theta)x} \right]_0^\infty = \frac{1}{1 - t\theta}, \qquad t < \frac{1}{\theta}.$$

$$M'(t) = \frac{\theta}{(1-\theta t)^2}, \qquad M''(t) = \frac{2\theta^2}{(1-\theta t)^3} \Rightarrow M'(0) = \theta, \qquad M''(0) = E(X^2) = 2\theta^2.$$

$$\Rightarrow \boxed{E(X) = \theta} \qquad Var(X) = E(X^2) - [E(X)]^2 = \theta^2.$$

mean waiting time

Example 1

Customers arrive in a certain shop according to APP at a mean rate of 20 per hour. What's the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

Solution:

Let X denote the waiting time in minutes until the first customer arrives, and note that $\lambda = 1/3$ is the mean number of arrivals per minute. Thus,

$$\theta = 1/\lambda = 3 \text{ and } f(x) = \frac{1}{3} e^{-\frac{1}{3}x}, \qquad x \geq 0.$$

Hence $P(X > 5) = \int_5^{\infty} \frac{1}{3} e^{-\frac{1}{3}x} dx = e^{-\frac{5}{3}}.$

Consider APP with mean λ in a unit interval,
Let W denote the waiting time until the αth occurrence.

Idea:
① Derive *cdf* of W: $F(w)$.
② $f(w)=\dfrac{d[F(w)]}{dw}$

- pdf of W

For $w \geq 0,\ F(w) = P(W \leq w) = 1 - P(W > w)$.

where $P(W > w) = P(\text{number of occurrences in } [0, w] \text{ smaller than } \alpha)$

$$= \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!}.$$

since the number of occurrences in the interval $[0,w]$ has a Poisson distribution with mean λw.

Therefore, $F(w) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!}$ for $w \geq 0$

➢ pdf of *W* (c.n.t.)

Since *W* is a **continuous** RV, $\frac{d[F(w)]}{dw}$, if exists, is **equal** to the pdf of *W*.

When $w > 0$, we have

$$F'(w) = -\left[\frac{(\lambda w)^0 e^{-\lambda w}}{0!}\right]' - \left[\sum_{k=1}^{\alpha-1}\frac{(\lambda w)^k e^{-\lambda w}}{k!}\right]' = \lambda e^{-\lambda w} - \sum_{k=1}^{\alpha-1}\left[\frac{k(\lambda w)^{k-1}\lambda}{k!}e^{-\lambda w} - \frac{(\lambda w)^k \lambda}{k!}e^{-\lambda w}\right]$$

$$= \lambda e^{-\lambda w} - e^{-\lambda w}\sum_{k=1}^{\alpha-1}\left[\frac{k(\lambda w)^{k-1}\lambda}{k!} - \frac{(\lambda w)^k \lambda}{k!}\right] = \lambda e^{-\lambda w} - e^{-\lambda w}\sum_{k=1}^{\alpha-1}\left[\frac{(\lambda w)^{k-1}\lambda}{(k-1)!} - \frac{(\lambda w)^k \lambda}{k!}\right]$$

$$= \lambda e^{-\lambda w} - e^{-\lambda w}\left[\sum_{k=1}^{\alpha-1}\frac{(\lambda w)^{k-1}\lambda}{(k-1)!} - \sum_{k=1}^{\alpha-1}\frac{(\lambda w)^k \lambda}{k!}\right] = \lambda e^{-\lambda w} - e^{-\lambda w}\left[\lambda - \frac{(\lambda w)^{\alpha-1}\lambda}{(\alpha-1)!}\right]$$

$$= \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!}e^{-\lambda w}.$$

If $w < 0$, then $F(w) = 0$ and $F'(w) = 0$.

A pdf of this form is said to be of the gamma type, and *W* is said to have a **gamma distribution**.

Definition [Gamma function]

$$\Gamma(t) = \int_0^{+\infty} y^{t-1} e^{-y} dy, \qquad t > 0.$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma(1) = \Gamma(2) = 1,$$

$$And \ \ for\ n \geq 2, \quad \Gamma(n) = (n-1)\Gamma(n-1).$$

The last statement is proved by induction on n. It's easy to see that, $\Gamma(1) = 1$. For $n \geq 2$, we will use integration by parts.

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ is due to the definite integration $\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, but we don't need to know how to derive it now.

Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Write $\Gamma(n) = \int_0^{\infty} f(x)g'(x)dx$, where, $f(x) = x^{n-1}$ and $g'(x) = e^{-x}$. Thus,

$$\Gamma(n) = \left[f(x)g(x) \right]_{x=0}^{\infty} + \int_{x=0}^{\infty} (n-1)x^{n-2}e^{-x}dx = (n-1)\Gamma(n-2),$$

as claimed.

Definition [Gamma distribution]

A *RV X* has a Gamma distribution if its pdf is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}}, \quad x \geq 0.$$

Accordingly, W, the waiting time until the αth occurrence in the APP, has a Gamma distribution with **parameters** $\boldsymbol{\alpha}$ and $\boldsymbol{\theta = \frac{1}{\lambda}}$.

- Gamma pdf f(x) is a well-defined pdf

Note that $f(x) \geq 0$

And $\int_{-\infty}^{+\infty} f(x)dx = \int_0^{\infty} \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^{\alpha}} dx,$

which, by change of variables with $y = x/\theta$, we have

$$\int_0^{\infty} \frac{(\theta y)^{\alpha-1}e^{-y}}{\Gamma(\alpha)\theta^{\alpha}} \theta dy = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} y^{\alpha-1}e^{-y} dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

➢Mean and Variance

The *mgf* of a Gamma distribution RV X is

$$M(t)=E(e^{tX})=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{\infty}e^{tx}x^{\alpha-1}e^{-x/\theta}dx$$

$$=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{\infty}x^{\alpha-1}e^{-(\frac{1}{\theta}-t)x}dx$$

$$=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{\infty}x^{\alpha-1}e^{-x\Big/(\frac{\theta}{1-\theta t})}dx. \qquad (\Theta)$$

Now we construct another gamma pdf!

$g(x)=\dfrac{1}{\Gamma(a)b^{a}}x^{a-1}e^{-x/b}$ is a pdf, we have that

$$\int_0^{\infty}\frac{1}{\Gamma(a)b^{a}}x^{a-1}e^{-x/b}=1$$

$$\Rightarrow\int_0^{\infty}x^{a-1}e^{-x/b}=\Gamma(a)b^{a} \qquad (\Upsilon)$$

Applying equation (Υ) to (Θ), with $b=\dfrac{\theta}{1-\theta t}$ and $a=\alpha$, we have:

$$M(t)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\Gamma(\alpha)(\frac{\theta}{1-\theta t})^{\alpha}=(\frac{1}{1-\theta t})^{\alpha} \quad if\ \ t<\frac{1}{\theta}.$$

➢Mean and Variance (c.n.t.)

$$M'(t) = \alpha\left(\frac{1}{1-\theta t}\right)^{\alpha-1}\left[-\frac{-\theta}{(1-\theta t)^2}\right] = \frac{\alpha\theta}{(1-\theta t)^{\alpha+1}}.$$

$$M''(t) = \frac{\alpha(\alpha+1)\theta^2}{(1-\theta t)^{\alpha+2}}.$$

$$\Rightarrow M'(0) = \alpha\theta. \qquad M''(0) = \alpha(\alpha+1)\theta^2$$

$$\Rightarrow E(X) = \alpha\theta. \qquad Var(X) = \alpha(\alpha+1)\theta^2 - (\alpha\theta)^2 = \alpha\theta^2$$

A special case is that $\alpha = 1$, Gamma distribution reduces to exponential distribution. α can be *non-integer*!

Definition [Beta distribution]

Let $X1$ and $X2$ have **independent gamma distributions** with **parameters** α, θ and β, θ, respectively. Take X=X1/(X1 + X2).
A *RV X* has a **Beta distribution** if its pdf is defined by

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1.$$