ISyE 3770 Assignment 3: Continuous and Joint Probability Distributions

Due date: 11:59 PM, Tuesday, Feb 6, 2024.

Remark: For Q2, Q3, Q4, and Q5, you are allowed to use the computer to obtain results. However, please keep in mind that during mid-term or final exams, you are not allowed to use electronic devices but only use the Table provided in the textbook. So please make sure you are familiar with how to use the Table. Please round all results into **3** digits after the decimal number.

Question 1 (Moment Generating Function)**.** *If the moment generating function of a random variable W is*

$$M(t) = (1-2t)^{-20},$$

find the pdf, mean, and variance of W. *(11 points)*

Solution. One can identify that W follows chi-square distribution with degree of freedom $r = 40$. Hence, the pdf of W is

$$f(w) = \frac{1}{19! \cdot 2^{20}} \cdot x^{19} e^{-w/2}, \quad w > 0.$$

Its mean and variance are $\mu = 40$ and $\sigma^2 = 80$, respectively. □

Question 2 (χ^2 Distribution)**.** *Suppose $X \sim \chi^2(17)$, find*

(a) $P(X < 7.564)$*;* *(3 points)*

(b) $P(X > 27.59)$*;* *(2 points)*

(c) $P(6.408 < X < 27.59)$*;* *(2 points)*

(d) $\chi^2_{0.95}(17)$*, recall that* $\chi^2_p(r)$ *denotes a number such that* $P(X > \chi^2_p(r)) = p$ *for* $X \sim \chi^2(r)$*;* *(2 points)*

(e) $\chi^2_{0.025}(17)$*.* *(2 points)*

Solution. (a) $P(X < 7.564) = 0.025$;

(b) $P(X > 27.59) = 0.050$;

(c) $P(6.408 < X < 27.59) = 0.940$;

(d) $\chi^2_{0.95}(17) = 8.672$;

(e) $\chi^2_{0.025}(17) = 30.191$.

□

Question 3 (Standard Normal Distribution)**.** *If $Z \sim N(0, 1)$, find* *(a)* $P(0 \leq Z \leq 0.87)$. *(3 points)*

(b) $P(-2.64 \leq Z \leq 0)$. *(2 points)*

(c) $P(|Z| > 1.39)$. *(2 points)*

(d) *a number* c *such that* $P(|Z| \leq c) = 0.95$. *(2 points)*

(e) $z_{0.0125}, z_{0.05}, z_{0.025}$. *Recall that* z_α *is a number such that* $P(Z \geq z_\alpha) = \alpha$. *(2 points)*

Solution. (a) $P(0 \leq Z \leq 0.87) = 0.308$;

(b) $P(-2.64 \leq Z \leq 0) = 0.496$;

(c) $P(|Z| > 1.39) = 0.165$;

(d) $c = 1.960$;

(e) $z_{0.0125} = 2.241, z_{0.05} = 1.645, z_{0.025} = 1.960$.

□

Question 4 (Normal Distribution)**.** *If* $Z \sim N(4, 25)$, *find*

(a) $P(4 \leq X \leq 10)$. *(3 points)*

(b) $P(-2 \leq X \leq 6)$. *(3 points)*

(c) $P(-4 \leq X \leq -2)$. *(3 points)*

(d) $P(X > 19)$. *(3 points)*

Solution. (a) $P(4 \leq X \leq 10) = 0.385$;

(b) $P(-2 \leq X \leq 6) = 0.540$;

(c) $P(-4 \leq X \leq -2) = 0.060$;

(d) $P(X > 19) = 0.001$.

□

Question 5 (Normal Distribution)**.** *If the moment generating function of* X *is* $M(t) = \exp(166t + 200t^2)$, *find*

(a) Mean of X. *(3 points)*

(b) Variance of X. *(3 points)*

(c) $P(170 \leq X \leq 200)$. *(3 points)*

(d) $P(148 \leq X \leq 172)$. *(3 points)*

Solution. (a,b) $\mu = 166, \sigma^2 = 400$;

(c) $P(170 \leq X \leq 200) = 0.376$;

(d) $P(148 \leq X \leq 172) = 0.434$.

□

Question 6 (Linear Transformation of Normal Distribution)**.** *Suppose* $X \sim N(\mu, \sigma^2)$, *show that the distribution of* $Y = aX + b$ *is* $N(a\mu + b, a^2\sigma^2)$, *where* $a \neq 0$. *(11 points)*

Hint: *Find the cdf* $P(Y \leq y)$ *of* Y, *and in the resulting integral, take* $w = ax + b$, *i.e.,* $x = (w - b)/a$.

Proof. The idea is to show Y has the same cdf as $N(a\mu+b, a^2\sigma^2)$.

$$
\begin{aligned}
P(Y \le y) = P(aX+b \le y) &= P(X \le (y-b)/a) \\
&= \int_{-\infty}^{(y-b)/a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(a\sigma)^2}} \exp\left(-\frac{(w-(b+a\mu))^2}{2(a\sigma)^2}\right) \mathrm{d}w,
\end{aligned}
$$

where the last equality is by the change of variable with $w = ax+b$. Therefore, $P(Y \le y)$ shares the same cdf as $N(a\mu+b, a^2\sigma^2)$, indicating $Y \sim N(a\mu+b, a^2\sigma^2)$. □

Question 7 (Joint pmf)**.** *Let the joint pmf of X and Y given by*

$$f(x,y) = \frac{x+y}{32}, x = 1,2, y = 1,2,3,4.$$

- *(a) Find $f_X(x)$, the marginal pmf of X;* *(3 points)*
- *(b) Find $f_Y(y)$, the marginal pmf of Y;* *(3 points)*
- *(c) Find $P(X > Y)$;* *(3 points)*
- *(d) Find $P(Y = 2X)$;* *(3 points)*
- *(e) Find $P(X + Y = 3)$.* *(3 points)*
- *(f) Find $P(X \le 3 - Y)$.* *(3 points)*
- *(g) Are X and Y independent or dependent? Why?* *(3 points)*
- *(h) Find the means and the variances of X and Y.* *(3 points)*

Solution. (a) $f_X(x) = \frac{2x+5}{16}, x = 1,2$;

(b) $f_Y(y) = \frac{2y+3}{32}, y = 1,2,3,4$;

(c) $P(X > Y) = \frac{3}{32}$;

(d) $P(Y = 2X) = \frac{9}{32}$;

(e) $P(X + Y = 3) = \frac{3}{16}$;

(f) $P(X \le 3 - Y) = \frac{1}{4}$;

(g) Since $f(x,y) \ne f_X(x) f_Y(y)$, they are dependent.

(h) $\mathbb{E}[X] = \frac{25}{16}, \text{Var}(X) = \frac{63}{256}, \mathbb{E}[Y] = \frac{45}{16}, \text{Var}(Y) = \frac{295}{256}$.

□

Question 8 (Correlation)**.** *Roll a fair four-sided dice twice. Let X equal the outcome on the first roll, and Y equal the sum of the two rolls. Determine $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, Cov$(X,Y)$, and ρ.* *(8 points)*

Soluution. The joint pmf of (X,Y) is

$$f(x,y) = \frac{1}{16}, x = 1,2,3,4, y = x+1, \dots, x+4.$$

Hence, $f_X(x) = 1/4, x = 1, 2, 3, 4$, which implies $\mu_X = \frac{5}{2}, \sigma_X^2 = \frac{5}{4}$. Also,

$$f_Y(y) = \begin{cases} 1/16, & \text{if } y = 2, 8 \\ 1/8, & \text{if } y = 3, 7 \\ 3/16, & \text{if } y = 4, 6 \\ 1/4, & \text{if } y = 5, \end{cases}$$

which implies $\mu_Y = 5$ and $\sigma_Y^2 = 5/2$. Finally,

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mu_X \mu_Y = \frac{5}{4}$$

and

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\sqrt{2}}{2}.$$

□