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Statistics and Applications

Introduction to Discrete Distribution

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Chapter 2.1 Discrete Distribution

Starting from this section, some typical random experiments and corresponding distribution will be introduced.

Section 2.4 ***Binomial distribution***

- Bernoulli experiment

  The outcome can be classified in one of two ***mutually exclusive and exhaustive*** ways--say either *success* or *failure*. (e.g. *female* or *male*; *life* or *death*)

- Bernoulli trials

  When a Bernoulli experiment is performed several *independent* times and the probability of *success*—say, p—remains the *same* from trial to trial. In other words, we let p donate the probability of *success* on each trial. And we define $q \triangleq 1 - p$ to donate the probability of *failure*.

Example 1:

You are a fan of lottery. For a lottery, the probability of winning is $\frac{1}{1000}$. If you buy the lottery for 10 successive days, that corresponds to 10 *Bernoulli trials* with $p=\frac{1}{1000}$.

Assuming independence

- Bernoulli distribution
  - Let X be a RV associated with a Bernoulli trial with the probability of success p.
  - Define RV

$$X : S \to X(S) \subseteq \mathbb{R}, \qquad S = \{\text{Success}, \text{Failure}\}$$

$$X(\text{Success}) = 1, X(\text{Failure}) = 0, \qquad X(S) = \{0, 1\}$$

  - The pmf of X can be written as:

$$f : X(S) = \{0, 1\} \to [0, 1],$$

$$x \mapsto f(x) = p^x (1-p)^{1-x}.$$

  - The RV X has a **Bernoulli distribution** with the following characteristic:

$$\mathbb{E}[X] = p, \quad \mathbb{V}\text{ar}[X] = pq, \quad M(t) = \mathbb{E}[e^{tX}] = (1-p) + p \cdot e^t.$$

- In a sequence of n Bernoulli trials, we shall let X_i denote the Bernoulli random variable associated with the i-th trial. An observed sequence of n Bernoulli trials will then be an n-tuple of zeros and ones, and we often call this *collection* a **random sample** of size n from a Bernoulli distribution

Example 2

Out of millions of instant lottery tickets, suppose that 20% are winners. If 5 tickets are purchased, then $(0, 0, 0, 1, 0)$ is a ***random sample***. Assuming *independence* between purchasing different tickets, the probability of this sample is

$$p = (0.2)(0.8)^4.$$

Multiplication Rule:
Suppose events $A_1, A_2, \ldots, A_n$ are *mutually independent*,

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n).$$

Binomial Distribution

- Motivation: We are interested in the *number of successes* in n Bernoulli trials, the order of the occurrence is not concerned.

A binomial experiment satisfies the following properties:

1. A Bernoulli experiment (i.e., Success & Failure) is performed n times.
2. Trials are independent.
3. Probability of success on each trial is a constant p; the probability of failure is $q = 1 - p$.
4. Define RV X as # of successes in n trials.

Multiplication rule of probability: $P(A \cap B) = P(A)P(B)$

Remark: $f(x)$ refers to binomial probability, X is said to have a Binomial distribution, denoted as $X \sim b(n, p)$

- $X : S \to X(S) = \{0, 1, \ldots, n\}$.
- When $x \in X(S)$, # of ways of selecting x successes in n trials is $\binom{n}{x}$.
- Since trials are independent, the probability of each way is $p(1-p)^{n-x}$.
- pmf of X:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n.$$

Why *pmf* is well-defined?

Example 2 (revisited)

Let the probability of producing a winning ticket to be 20%. If X is the number of winning tickets, where $n = 5$ tickets are purchased, then the probability of purchasing 2 winning tickets is

$$f(2) = P(X = 2) = \binom{5}{2}(0.2)^2(0.8)^3, \quad X \sim b(5, 0.2).$$

➢cdf of Binomial distribution

- Assume X have a Binomial distribution $b(n, p)$, the cdf of X is

$$F(x) = P(X \leq x) = \sum_{y \in X(S):\ y \leq x} f(y) = \sum_{y=0}^{\lfloor x \rfloor} \binom{n}{y} p^y (1-p)^{n-y}$$

Here $x \in (-\infty, \infty)$, and $\lfloor x \rfloor$ denotes the largest integer that is no more than x.

- Assume X have a Binomial distribution $b(n,p)$, the cdf of X is

$$F(x) = P(X \le x) = \sum_{y \in X(S):\ y \le x} f(y) = \sum_{y=0}^{\lfloor x \rfloor} \binom{n}{y} p^y (1-p)^{n-y}$$

Example 3

Chickens are raised for laying eggs. Let $p = 0.5$ be the probability that the newly hatched chick is a female. Assuming independence, let X be the number of female chicks out of 10 newly hatched chicks selected at random.

- Obviously, $X \sim b(10,0,5)$. Compute

$$P(X \le 5), P(X = 6), P(X \ge 6).$$

Solution:

$$P(X \le 5) = \sum_{x=0}^{5} \binom{10}{x} (0.5)^x (0.5)^{10-x}, \qquad P(X \ge 6) = 1 - P(X \le 5).$$

$$P(X = 6) = f(6) = \binom{10}{x} (0.5)^6 (0.5)^4.$$

➢mgf of Binomial distribution

Assume X have a binomial distribution $b(n,p)$, the mgf of X is

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= \left[(1-p) + pe^t\right]^n, \qquad t \in (-\infty, \infty).$$

From the expansion of

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x},$$

with $a = pe^t, b = 1-p$.

- $M'(t) = n\left[(1-p) + pe^t\right]^{n-1} pe^t$, which implies

$$M'(0) = \mathbb{E}[X] = np.$$

- $M''(t) = n(n-1)\left[(1-p) + pe^t\right]^{n-2} (pe^t)^2 + n\left[(1-p) + pe^t\right]^{n-1} pe^t$, which implies

$$M''(0) = \mathbb{E}[X^2] = n(n-1)p^2 + np.$$

- $\mathbb{V}\mathrm{ar}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = np(1-p).$

Besides, when $n = 1$, the ***Binomial distribution*** reduces to ***Bernoulli distribution***.

Section 2.5 *Negative Binomial distribution*

- Motivation: We are interested in the situation that we observe a sequence of independent Bernoulli trials until exactly r successes occur, where r is a fixed positive integer.
- Define RV X to be the trial number, on which the r-th success is observed.

$$X: \; S \to X(S) = \{r, r+1, \ldots\}$$

- Let $f(x)$ denote the pmf of X.

$$f(x) = P(\{\text{At the } x\text{-th trial, } r\text{-th success is observed}\})$$

$$= P\Big(\underbrace{\Big\{\text{for the first } x-1 \text{ trials, } r-1 \text{ success have been observed}\Big\}}_{\text{def } A} \bigcap \underbrace{\Big\{\text{At the } x\text{-th trial, the outcome is success}\Big\}}_{\text{def } B}\Big)$$

def A

def B

$$P(A) = \binom{x-1}{r-1} p^{r-1}(1-p)^{x-r}, \qquad P(B) = p.$$

$$\Longrightarrow \; f(x) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r}, \quad x = r, r+1, \ldots$$

Special Case of Negative Binomial Distribution for $r = 1$

- $f(x) = p(1-p)^{x-1}$.
- For fixed $k \in \mathbb{N}_+$,

$$P(X > k) = \sum_{x=k+1}^{\infty} p(1-p)^{x-1} = \frac{p(1-p)^k}{1-(1-p)} = (1-p)^k.$$

$$P(X \le k) = 1 - P(X > k) = 1 - (1-p)^k$$

Example 1

- Biology students are checking eye color of fruit flies.
- For individual fly, P(white)=1/4, P(red)=3/4.
- Assume the observations are independent Bernoulli trials.

At least 4 Flies: $P(X \ge 4) = P(X > 3) = (1 - 1/4)^3 = (3/4)^3$

At most 4 Flies: $P(X \le 4) = 1 - (1 - 1/4)^4$

4 Flies: $P(X = 4) = (1/4)(3/4)^3$

➢Mean and Variance

Prove the following for X having a negative binomial distribution:

$$\mathbb{E}[X] = \frac{r}{p}, \qquad \mathbb{V}\mathrm{ar}[X] = \frac{r(1-p)}{p^2}.$$

- *Direct Calculation*
- *Using mgf*

Proof: The mgf of X is

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=r}^{\infty} e^{tx} \binom{x-1}{r-1} p^r (1-p)^{x-r} = (pe^t)^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} \left[(1-p)e^t\right]^{x-r}$$

$$= \frac{(pe^t)^r}{\left[1-(1-p)e^t\right]^r}, \qquad \text{where } (1-p)e^t < 1.$$

Therefore,

$$M'(t) = r(pe^t)^r \left[1-(1-p)e^t\right]^{-r-1},$$

$$M''(t) = r(pe^t)^r(-r-1)\left[1-(1-p)e^t\right]^{-r-2}\left[-(1-p)e^t\right]$$
$$+ r^2(pe^t)^{r-1}(pe^t)\left[1-(1-p)e^t\right]^{-r-1}.$$

$$\Longrightarrow \mathbb{E}[X] = M'(0) = rp^{-1}, \qquad \mathbb{E}[X^2] = M''(0) = rp^{-2}(r+1-p).$$
$$\Longrightarrow \mathbb{V}\mathrm{ar}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = rp^{-2}(1-p).$$

Section 2.6 *Poisson Distribution*

There are experiments that result in counting the number of times particular events occur at given times or with given physical objects.

E.g.

- The number of flaws in a 100 feet long wire
- The number of customers that arrive at a ticket window between 9p.m. to 10p.m.

Counting such events can be looked upon as observations of a ***random variable*** associated with an ***approximate Poisson process(APP)***, provided that the conditions in the following definition are satisfied.

Definition [Approximate Possion Process (App)]

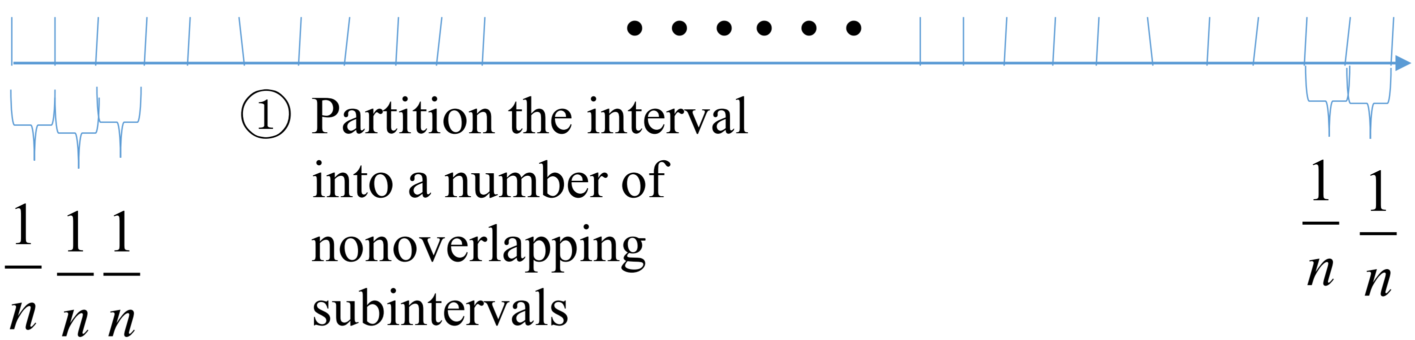
Let the number of occurrences of some event in a given continuous interval be counted. Then we have an **APP** with parameter $\lambda \geq 0$ if

(a) The numbers of occurrences in nonoverlapping subintervals are **independent**.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.

(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially **0**.

Consider a random experiment desired by App. Let X denote the number of occurrences in an interval of length 1. We aim to find an approximation for $P(X = x)$, where x is a nonnegative integer.



② If n is sufficiently large ($n \gg x$), $P(X = x)$ can be approximated by the probability that exactly x of these n subintervals each has one occurrence.

③ I. By condition (b), the probability of one occurrence in anyone subinterval of length $1/n$ is approximately λ/n.

II. By condition (c), the probability of 2 or more occurrences in any one subinterval is essentially 0. That is, For each subinterval there is either no occurrence or one occurrence. [The probability of occurrence is $\frac{\lambda}{n}$.

Conditions I and II implies that the occurrence and non-occurrence in each interval can be treated as Bernoulli trials.

III. By condition (a), we have a sequence of n Bernoulli trials with probability p approximately equal to $\frac{\lambda}{n}$.

Number of Occurrence follows $b(1, \frac{\lambda}{n})$

④ Therefore, $P(X = x)$ can be approximated by the binomial probability:

$$P(X=x)\approx\frac{n!}{x!(n-x)!}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x}$$

⑤ If let $n\to\infty$, then

$$\lim_{n\to\infty}\frac{n!}{x!(n-x)!}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x}=\lim_{n\to\infty}\frac{n!}{(n-x)!n^x}\cdot\frac{\lambda^x}{x!}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-x}$$

Now for fixed n, we have:

$$\lim_{n\to\infty}\frac{n!}{(n-x)!n^x}=\lim_{n\to\infty}\frac{n(n-1)\cdots(n-x+1)}{n^x}=\lim_{n\to\infty}\left[1\cdot(1-\frac{1}{n})\cdots(1-\frac{x-1}{n})\right]=1,$$

$$\lim_{n\to\infty}(1-\frac{\lambda}{n})^n=e^{-\lambda},\lim_{n\to\infty}(1-\frac{\lambda}{n})^{-x}=1$$

We have $P(X=x)=\lim_{n\to\infty}\frac{n!}{(n-x)!n^x}\cdot\frac{\lambda^x}{x!}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-x}=\frac{\lambda^x e^{-\lambda}}{x!}.$

Since we know $\lim_{n\to\infty}(1+\frac{x}{n})^n=e^x$,

we replace x with $-\lambda$.

Definition 2.6-2 [Poisson distribution]

It can be verified that

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, \ldots$$

is a well-defined pmf. If a RV X has $f(x)$ as its *pmf*, then X is said to have a **Poisson distribution**.

What is the interpretation of λ ?

- Mean and Variance

The mgf of a Poisson distribution for a RV X is

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.$$

λ is the average number, or variance of occurrences in the interval.

$$M'(t) = \lambda e^t e^{\lambda(e^t-1)} \implies M'(0) = \lambda = \mathbb{E}[X]$$

$$M''(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)} \implies M''(0) = \lambda + \lambda^2 = \mathbb{E}[X^2]$$

$$\mathbb{V}\mathrm{ar}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda$$

Example 1

In a large city, telephone calls to 110 come on the average of 2 every 3 minutes. If one models with App, what is the probability of five or more calls arriving in a 9-minute period?

Solution. Let X denote the number of calls in a 9-min period. Then $\mathbb{E}[X] = 6 = \lambda$, which implies $f(x) = \frac{6^x e^{-6}}{x!}$.
Hence, $P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^{4} \frac{6^x e^{-6}}{x!}$.