ISyE 3770 Assignment 2: Discrete and Continuous Distributions

Due date: 11:59 PM, Tuesday, Feb 6, 2024.

Question 1 (Mathematical Expectation May not Exist!). *Let the pmf of X be defined by $f(x) = \frac{6}{\pi^2 x^2}, x = 1, 2, \ldots$. Show that $\mathbb{E}[X] = +\infty$, and thus, does not exist.* *(11 points)*

Solution. Check that

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} x \cdot \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x},$$

which diverges to infinity. □

Question 2 (mgf). *For the following moment generating function of a random variable X, (i) Give the name of the distribution of X (if it has a name), (ii) find the values of mean and variance, (iii) calculate $P(1 \le X \le 2)$:*

(a) $M(t) = (0.3 + 0.7e^t)^5$; *(3 points)*

(b) $M(t) = \frac{0.3e^t}{1-0.7e^t}, \quad t < -\ln(0.7)$; *(2 points)*

(c) $M(t) = 0.45 + 0.55e^t$; *(2 points)*

(d) $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t}$; *(2 points)*

(e) $M(t) = \sum_{x=1}^{10} (0.1)e^{tx}$. *(2 points)*

Solution. (a) Binomial distribution, $X \sim b(5, 0.7)$.

$\mu = np = 3.5$ and $\sigma^2 = np(1-p) = 1.05$.

$P(1 \le X \le 2) = 0.1607$.

(b) Geometric distribution with $p = 0.3$.

$\mu = 1/p = 10/3$ and $\sigma^2 = (1-p)/p^2 = 70/9$.

$P(1 \le X \le 2) = 0.51$.

(c) Bernoulli distribution with $p = 0.55$.

$\mu = p = 0.55$ and $\sigma^2 = p(1-p) = 0.2475$.

$P(1 \le X \le 2) = 0.55$.

(d) Discrete distribution (please do not deduct points if students do not give its name).

$\mu = 2.1$ and $\sigma^2 = 0.89$.

$P(1 \le X \le 2) = 0.7$.

(e) Discrete uniform distribution (please do not deduct points if students do not give its name).

$\mu = 5.5$ and $\sigma^2 = 8.25$.

$P(1 \le X \le 2) = 0.2$.

□

Question 3 (Binomial Distribution)**.** *In a lab experiment involving inorganic syntheses of molecular precursors to organometallic ceramics, the final step of a five-step reaction involves the formation of a metal-metal bond. The probability of such a bond forming is* $p = 0.20$*. Let* X *equal the number of successful reactions out of* $n = 25$ *experiments.*

(a) Find the probability that X *is at most* 4*.* *(2 points)*

(b) Find the probability that X *is at least* 5*.* *(3 points)*

(c) Find the probability that X *is equal to* 6*.* *(3 points)*

(d) Give the mean, variance, and standard deviation of X*.* *(3 points)*

Solution. Check that $X \sim b(25, 0.2)$.

(a) $P(X \leq 4) = 0.42$.

(b) $P(X \leq 5) = 0.58$.

(c) $P(X = 6) = 0.16$.

(d) $\mu = np = 5$, $\sigma^2 = np(1-p) = 4$, and $\sigma = 2$.

□

Question 4 (Variant of Geometric Distribution)**.** *Let* X *equal the number of flips of a fair coin that are required to observe the same face on consecutive flips.*

(a) Find the pmf of X*.* *(3 points)*

(b) Find the mgf of X*.* *(3 points)*

(c) Use the mgf to find the values of the mean and variance of X*.* *(3 points)*

(d) Find the value of $P(X \leq 3), P(X \geq 5)$*, and* $P(X = 3)$*.* *(3 points)*

Solution. (a) Suppose x is the number of flips such that we observe the same face on consecutive flips. Then it holds that (i) heads and tails must alternate in the first $(x-1)$ flips; (ii) x-th flip matches $(x-1)$-th flip. The probability that event (i) happens equals $2 \cdot \frac{1}{2^{x-1}}$, since there are only two possible outcomes: HTHTHT. . ., or THTHTH. . .. The probability that event (ii) happens equals $\frac{1}{2}$. Multiplying those two probability terms, we obtain the pmf:

$$f(x) = P(X = x) = \frac{1}{2^{x-1}}, \qquad x = 2, 3, \ldots.$$

(b) $M(t) = \frac{e^{2t}}{2-e^t}, \quad t < \ln 2$.

(c) $\mu = 3$ and $\sigma^2 = 2$.

(d) $P(X \leq 3) = 3/4, P(X \geq 5) = 1/8, P(X = 3) = 1/4$.

□

Question 5 (Poisson Distribution)**.** *Flaws in a certain type of drapery material appear on the average of one in* 150 *square feet. If we assume a Poisson distribution, find the probability of at most one flaw appearing in* 225 *square feet.* *(11 points)*

Solution. $\lambda = \frac{225}{150} = 1.5$, and X follows Poisson distribution with parameter λ. Hence,

$$P(X \le 1) = 0.558.$$

□

Question 6 (Continuous Random Variable)**.** *For each of the following functions, (i) find the constant c such that $f(x)$ is a pdf of a random variable X; (ii) find the cdf $F(x) = P(X \le x)$; (iii) plot graphs of the pdf $f(x)$ and the distribution function $F(x)$, and (iv) find mean and variance:*

(a) $f(x) = x^3/4, 0 < x < c$. *(3 points)*

(b) $f(x) = (3/16)x^2, -c < x < c$. *(4 points)*

(c) $f(x) = c/\sqrt{x}, 0 < x < 1$. *Is this pdf bounded?* *(4 points)*

Solution. (a) $c = 2$,

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^4/16, & \text{if } 0 \le x \le 2 \\ 1, & \text{if } x > 2. \end{cases}$$

$\mu = 8/5$ and $\sigma^2 = 8/75$.

(b) $c = 2$,

$$F(x) = \begin{cases} 0, & \text{if } x < -2 \\ x^3/16 + 1/2, & \text{if } -2 \le x \le 2 \\ 1, & \text{if } x > 2. \end{cases}$$

$\mu = 0$ and $\sigma^2 = 12/5$.

(c) $c = 1/2$,

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1. \end{cases}$$

$\mu = 1/3$ and $\sigma^2 = 4/45$.

□

Question 7 (Uniform Distribution)**.** *Customers arrive randomly at a bank tellers window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minutes that the customer arrived. If X is $U(0, 10)$, find:*

(a) The pdf of X; *(3 points)*

(b) $P(X \ge 8)$*;* *(2 points)*

(c) $P(2 \le X < 8)$*;* *(2 points)*

(d) $\mathbb{E}[X]$*;* *(2 points)*

(e) $\mathrm{Var}(X)$. *(2 points)*

Solution. (a) $f(x) = 1/10$ for $0 \le x \le 10$.

(b) $P(X \ge 8) = 0.2$

(c) $P(2 \le X \le 8) = 0.6$

(d) $\mathbb{E}[X] = 5$

(e) $\mathrm{Var}(X) = 25/3$

□

Question 8 (Percentile). *Consider the pdf* $f(x) = (x+1)/2, -1 < x < 1$. *Find*

(a) $\pi_{0.64}$; *(3 points)*

(b) $q_1 \triangleq \pi_{0.25}$; *(4 points)*

(c) $\pi_{0.81}$. *(4 points)*

Solution. (a) $\pi_{0.64} = 0.6$

(b) $\pi_{0.25} = 0$

(c) $\pi_{0.81} = 0.8$

□

Question 9 (Exponential Distribution). *Let* X *have an exponential distribution with mean* $\Theta > 0$. *Show that*

$$P(X > x + y \mid X > x) = P(X > y), \quad \forall x, y > 0. \qquad \text{(11 points)}$$

Proof. Check that

$$P(X > x + y \mid X > x) = \frac{P(X > x + y)}{P(X > x)} = \frac{e^{-(x+y)/\Theta}}{e^{-x/\Theta}} = e^{-y/\Theta} = P(X > y).$$

□