ISyE 3770 Assignment 2: Discrete and Continuous Distributions

Due date: 11:59 PM, Tuesday, Feb 6, 2024.

Question 1 (Mathematical Expectation May not Exist!). *Let the pmf of X be defined by $f(x) = \frac{6}{\pi^2 x^2}, x = 1, 2, \ldots$. Show that $\mathbb{E}[X] = +\infty$, and thus, does not exist.* *(11 points)*

Question 2 (mgf). *For the following moment generating function of a random variable X, (i) Give the name of the distribution of X (if it has a name), (ii) find the values of mean and variance, (iii) calculate $P(1 \leq X \leq 2)$:*

- *(a)* $M(t) = (0.3 + 0.7e^t)^5$; *(3 points)*
- *(b)* $M(t) = \frac{0.3e^t}{1-0.7e^t}, \quad t < -\ln(0.7)$; *(2 points)*
- *(c)* $M(t) = 0.45 + 0.55e^t$; *(2 points)*
- *(d)* $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t}$; *(2 points)*
- *(e)* $M(t) = \sum_{x=1}^{10}(0.1)e^{tx}$. *(2 points)*

Question 3 (Binomial Distribution). *In a lab experiment involving inorganic syntheses of molecular precursors to organometallic ceramics, the final step of a five-step reaction involves the formation of a metal-metal bond. The probability of such a bond forming is $p = 0.20$. Let X equal the number of successful reactions out of $n = 25$ experiments.*

- *(a) Find the probability that X is at most 4.* *(2 points)*
- *(b) Find the probability that X is at least 5.* *(3 points)*
- *(c) Find the probability that X is equal to 6.* *(3 points)*
- *(d) Give the mean, variance, and standard deviation of X.* *(3 points)*

Question 4 (Variant of Geometric Distribution). *Let X equal the number of flips of a fair coin that are required to observe the same face on consecutive flips.*

- *(a) Find the pmf of X.* *(3 points)*
- *(b) Find the mgf of X.* *(3 points)*
- *(c) Use the mgf to find the values of the mean and variance of X.* *(3 points)*
- *(d) Find the value of $P(X \leq 3), P(X \geq 5)$, and $P(X = 3)$.* *(3 points)*

Question 5 (Poisson Distribution). *Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume a Poisson distribution, find the probability of at most one flaw appearing in 225 square feet.* *(11 points)*

Question 6 (Continuous Random Variable)**.** *For each of the following functions, (i) find the constant* c *such that* $f(x)$ *is a pdf of a random variable* X*; (ii) find the cdf* $F(x) = P(X \leq x)$*; (iii) plot graphs of the pdf* $f(x)$ *and the distribution function* $F(x)$*, and (iv) find mean and variance:*

(a) $f(x) = x^3/4, 0 < x < c$. *(3 points)*

(b) $f(x) = (3/16)x^2, -c < x < c$. *(4 points)*

(c) $f(x) = c/\sqrt{x}, 0 < x < 1$. *Is this pdf bounded?* *(4 points)*

Question 7 (Uniform Distribution)**.** *Customers arrive randomly at a bank tellers window. Given that one customer arrived during a particular 10-minute period, let* X *equal the time within the 10 minutes that the customer arrived. If* X *is* $U(0, 10)$*, find:*

(a) The pdf of X*;* *(3 points)*

(b) $P(X \geq 8)$*;* *(2 points)*

(c) $P(2 \leq X < 8)$*;* *(2 points)*

(d) $\mathbb{E}[X]$*;* *(2 points)*

(e) $\mathrm{Var}(X)$. *(2 points)*

Question 8 (Percentile)**.** *Consider the pdf* $f(x) = (x + 1)/2, -1 < x < 1$. *Find*

(a) $\pi_{0.64}$*;* *(3 points)*

(b) $q_1 \triangleq \pi_{0.25}$*;* *(4 points)*

(c) $\pi_{0.81}$. *(4 points)*

Question 9 (Exponential Distribution)**.** *Let* X *have an exponential distribution with mean* $\Theta > 0$. *Show that*

$$P(X > x + y \mid X > x) = P(X > y).$$ (11 points)