ISyE 3770 Assignment 1: Introduction to Probability and Random Variables

Due date: 11:59 PM, Tuesday, Jan 23, 2024.

Question 1 (Algebra of Sets)**.** *If* $P(A) = 0.5, P(B) = 0.5,$ *and* $P(A \cap B) = 0.3$*. Find* ***(a)*** $P(A \cup B)$, ***(b)*** $P(A \cap B')$, *and* ***(c)*** $P(A' \cup B')$.

Solution. It can be shown that

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) = 0.7, \\
P(A \cap B') &= P(A) - P(A \cap B) = 0.2, \\
P(A' \cup B') &= 1 - P(A \cap B) = 0.7.
\end{aligned}
$$

□

Question 2 (Properties of Probability)**.** *(a) Suppose* A, B, C *are any three events, prove that*

$$
\begin{aligned}
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \\
- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).
\end{aligned}
$$

(b) Roll a fair 6*-sided die three times. Let*

$$
\begin{aligned}
A_1 &= \{1 \text{ or } 2 \text{ on the first roll}\}, \\
A_2 &= \{3 \text{ or } 4 \text{ on the second roll}\}, \\
A_3 &= \{5 \text{ or } 6 \text{ on the third roll}\}.
\end{aligned}
$$

It is given that $P(A_i) = 1/3, i = 1, 2, 3$*;* $P(A_i \cap A_j) = (1/3)^2, i \neq j$*; and* $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$*. Use result in Part (a) to find* $P(A_1 \cup A_2 \cup A_3)$.

Solution. 1) Since $P(A \cup B \cup C) = A \cup (B \cup C)$, it holds that

$$
\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\
&= P(A) + \Big[P(B) + P(C) - P(B \cap C)\Big] - P((A \cap B) \cup (A \cap C)) \\
&= P(A) + \Big[P(B) + P(C) - P(B \cap C)\Big] - \Big[P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)\Big].
\end{aligned}
$$

Simplifying the relation above gives the desired result.

2) We can find

$$
P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i,j=1, i \neq j}^{3} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3) = \frac{1}{3} * 3 - (1/3)^2 * 3 + (1/3)^3 = \frac{19}{27}.
$$

□

Question 3 (Permutation and Combination)**.** *Three students (S) and six faculty members (F) are on a panel discussing a new college policy.*

(a) In how many different ways can the nine participants be lined up at a table in the front of the auditorium?

(b) How many lineups are possible, considering only the labels S and F?

(c) For each of the nine participants, you are to decide whether the participant did a good job or a poor job, stating his or her opinion of the new policy; that is, giving each of the nine participants a grade of G or P. How many different "scorecards" are possible?

Solution. 1) Since nine individuals are different, they are arranged in $9!$ ways.

Note: In Q3 and Q4, it is optional to simplify permutation and combination numbers. If students do not simplify them, please do not reduce points.

2) The arrangement between students and faculty members is distinguishable permutation, which has $\binom{9}{3} = 84$ ways.

3) Each participants could have grade of G or P. So there are $2^9 = 512$ outcomes.

□

Question 4 (Method of Enumeration)**.** *A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:*

(a) Four of a kind (four cards of equal face value and one card of a different value).

(b) Full house (one pair and one triple of cards with equal face value).

(c) Three of a kind (three equal face values plus two cards of different values).

(d) Two pairs (two pairs of equal face value plus one card of different value).

(e) One pair (one pair of equal face value plus three cards of different values).

Solution. 1) We will first pick 1 face value from A to K in 13 ways, and then pick one card in the remaining cards with different values. By multiplication rule,

$$P(\text{Four of a kind}) = \frac{\binom{13}{1}\binom{52-4}{1}}{\binom{52}{5}} = 0.00024.$$

2) We will first pick 1 face value to have 3 cards, and then pick another face value to have 2 cards.

$$P(\text{Full house}) = \frac{\left[\binom{13}{1}\binom{4}{3}\right]\left[\binom{13-1}{1}\binom{4}{2}\right]}{\binom{52}{5}} = 0.00144.$$

3) We will first pick 1 face value to have 3 cards and then another two face values to have 1 card, respectively.

$$P(\text{Three of a kind}) = \frac{\left[\binom{13}{1}\binom{4}{3}\right]\left[\binom{13-1}{2}\binom{4}{1}\binom{4}{1}\right]}{\binom{52}{5}} = 0.0211.$$

4) We will first pick 2 face values to have 2 cards, respectively, and then pick the last one from the remaining cards with a different face value.

$$P(\text{Two pairs}) = \frac{\left[\binom{13}{2}\binom{4}{2}\binom{4}{2}\right]\binom{52-4-4}{1}}{\binom{52}{5}} = 0.0475.$$

5) We will pick 1 face value to have 2 cards, and then pick three cards with different face values.

$$P(\text{One pairs}) = \frac{\left[\binom{13}{1}\binom{4}{2}\right]\binom{13-1}{3}\binom{4}{1}^3}{\binom{52}{5}} = 0.423.$$

□

Question 5 (Conditional Probability)**.** *In the gambling game craps, a pair of dice is rolled and the outcome of the experiment is the sum of the points on the up sides of the* 6*-sided dice. The bettor wins on the first roll if the sum is* 7 *or* 11*. The bettor loses on the first roll if the sum is* $2, 3,$ *or* 12*. If the sum is* $4, 5, 6, 8, 9,$ *or* 10*, that number is called the bettor's point. Once the point is established, the rule is as follows: If the bettor rolls a* 7 *before the point, the bettor loses, but if the point is rolled before a* 7*, the bettor wins.*

(a) List the 36 *outcomes in the sample space for the roll of a pair of dice.*

(b) Find the probability that the bettor wins on the first roll. That is, find the probability of rolling a 7 *or* 11*, denoted as* $P(7 \text{ or } 11)$*.*

(c) Given that 8 *is the outcome on the first roll, find the probability that the bettor now rolls the point* 8 *before rolling a* 7 *and thus wins. Note that at this stage in the game, the only outcomes of interest are* 7 *and* 8*. Thus find* $P(8 \mid 7 \text{ or } 8)$*.*

(d) The probability that a bettor rolls an 8 *on the first roll, and then wins is given by* $P(8)P(8 \mid 7 \text{ or } 8)$*. Show that this probability is* $(5/36)(5/11)$*.*

(e) Show that the total probability that a bettor wins in the game of craps is 0.49293*. Hint: Note that the bettor can win in one of the several mutually exclusive ways: by rolling a* 7 *or an* 11 *on the first roll, or by establishing one of the points* $4, 5, 6, 8, 9, 10$ *on the first roll and then obtaining that point on successive rolls before a* 7 *comes up.*

Solution.. 1) They could be

$$\begin{gathered}(1,1),(1,2),\ldots,(1,6)\\ \vdots\\ (6,1),(6,2),\ldots,(6,6)\end{gathered}$$

2) For the 36 outcomes listed above, they have 8 outcomes of rolling a 7 or 11:

$$(1,6),(2,5),(3,4),(4,3),(5,2),(5,6),(6,1),(6,5).$$

Hence this probability $P(\text{7 or 11}) = 8/36 = 2/9$.

3) It holds that

$$P(8 \mid \text{7 or 8}) = \frac{P(8 \cap \text{7 or 8})}{P(\text{7 or 8})} = \frac{5}{11}.$$

4) It suffices to verify that $P(8) = 5/36$.

5) There are two ways to get a bettor wins:

- Rolling a 7 or 11 on the first roll
- Rolling a point on the first roll and then rolling a 7 on successive rolls.

Those are mutually exclusive events, and therefore

$$P(\text{A bettor wins}) = 2/9 + 2 * (5/36 * 5/11 + 4/36 * 4/10 + 3/36 * 3/9) = 0.49293.$$

□

Question 6 (Bayes Theorem)**.** *Suppose we want to investigate the percentage of abused children in a certain population. To do this, doctors examine some of these children taken at random from that population. However, doctors are not perfect: They sometimes classify an abused child* (A^+) *as one not abused* (D^-)*, or they classify a nonabused child* (A^-) *as one that is abused* (D^+)*. Suppose these error rates are*

$$P(D^- \mid A^+) = 0.08, \quad P(D^+ \mid A^-) = 0.05.$$

Thus, $P(D^+ \mid A^+) = 0.92, \quad P(D^- \mid A^-) = 0.95$*. Let us pretend that only* 2% *of all children are abused, i.e.,* $P(A^+) = 0.02$ *and* $P(A^-) = 0.98$*.*

(a) Select a child at random. What is the probability that the doctor classifies this child as abused? That is, to compute $P(D^+) = P(A^+)P(D^+ \mid A^+) + P(A^-)P(D^+ \mid A^-)$*.*

(b) Compute $P(A^- \mid D^+)$ *and* $P(A^+ \mid D^+)$*.*

(c) Compute $P(A^- \mid D^-)$ *and* $P(A^+ \mid D^-)$*.*

(d) Are the probabilities in (b) and (c) alarming? This happens because the error rates of 0.08 and 0.05 are high relative to the fraction 0.02 of abused children in the population.

Solution. 1)

$$P(D^+) = P(A^+)P(D^+ \mid A^+) + P(A^-)P(D^+ \mid A^-) = 0.0674.$$

2)

$$P(A^- \mid D^+) = \frac{P(D^+ \mid A^-)P(A^-)}{P(D^+)} = 0.727$$
$$P(A^+ \mid D^+) = \frac{P(D^+ \mid A^+)P(A^+)}{P(D^+)} = 0.273$$

3)

$$P(A^- \mid D^-) = \frac{P(D^- \mid A^-)P(A^-)}{P(D^-)} = 0.998$$
$$P(A^+ \mid D^-) = \frac{P(D^- \mid A^+)P(A^+)}{P(D^-)} = 0.002$$

4) Yes. Particularly the high rates in those in part (b).

□

Question 7 (Introduction to Random Variable)**.** *For each of the following, determine the constant* c *so that* $f(x)$ *satisfies the conditions of being a pmf for a random variable* X*, and then depict each pmf as a line graph:*

(a) $f(x) = x/c, x = 1, 2, 3, 4.$

(b) $f(x) = cx, x = 1, 2, \ldots, 10.$

(c) $f(x) = c \cdot (1/4)^x, x = 1, 2, \ldots.$

(d) $f(x) = c \cdot (x+1)^2, x = 0, 1, 2, 3.$

(e) $f(x) = x/c, \quad x = 1, 2, \ldots, n.$

(f) $f(x) = \frac{c}{(x+1)(x+2)}, x = 0, 1, 2, \ldots.$ *Hint: write* $f(x) = \frac{c}{x+1} - \frac{c}{x+2}$.

Solution. Using the condition that $\sum_{x \in B} f(s) = 1$, it is easy to determine the value of c.

1) $c = 10$
2) $c = 1/55$
3) $c = 3$
4) $c = 1/30$
5) $c = \frac{n(n+1)}{2}$
6) $c = 1.$

Note: When I write $x = 0, 1, 2, ..., .$, it represents they are a sequence of countably infinite numbers. When I write $x = 0, 1, \ldots, n$, it represents they are a sequence of $n+1$ finite numbers. □

Question 8 (Expectation)**.** *For each of the following distributions, find* $\mu := \mathbb{E}[X]$, $\mathbb{E}[X(X-1)]$, *and* $\sigma^2 := \mathbb{E}[X(X-1)] + \mu - \mu^2$*:*

(a) $f(x) = \frac{3!}{x!(3-x)!}(\frac{1}{4})^x(\frac{3}{4})^{3-x}, x = 0, 1, 2, 3.$

(b) $f(x) = \frac{4!}{x!(4-x)!}(\frac{1}{2})^4, x = 0, 1, 2, 3, 4.$

Solution. 1)

$$\mu = \mathbb{E}[X] = \sum_{x=1}^{3} x \cdot \left[\frac{3!}{x!(3-x)!}(\frac{1}{4})^x(\frac{3}{4})^{3-x}\right] = \frac{3}{4}\sum_{k=0}^{2}\frac{2!}{k!(2-k)!}(\frac{1}{4})^k(\frac{3}{4})^{2-k} = \frac{3}{4}.$$

$$\mathbb{E}[X(X-1)] = \sum_{x=2}^{3} x(x-1) \cdot \left[\frac{3!}{x!(3-x)!}(\frac{1}{4})^x(\frac{3}{4})^{3-x}\right] = \frac{6}{16}(1/4 + 3/4) = \frac{6}{16} = \frac{3}{8}.$$

$$\sigma^2 = \mathbb{E}[X(X-1)] + \mu - \mu^2 = 9/16.$$

2) Similarly, $\mu = 2, \mathbb{E}[X(X-1)] = 3, \sigma^2 = 1.$

□