ISyE 3770, Spring 2024
Statistics and Applications

Confidence Intervals

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Motivation

- Engineers/data scientists often involved with estimating parameters
  - The number of customers browsing a webpage
  - Inside diameter of wheel
- Suppose you have tested 10 samples (users)
- You can use sample mean to estimate
- But the estimate can be *close* or very *far* from the true mean
- To avoid this, report the estimate for a range of plausible values called <u>confidence interval</u>

Confidence Statements

75.600001

- **Fortune Teller**

"I believe the answer is 75.6 meters"

- **Scientist**

[75.6−2.0, 75.6+2.0]

"I believe the answer is 75.6 meters plus or minus 2.0"

Two-sided
Confidence Interval

Confidence Level: in a nutshell

- **A confidence level specifies a confidence level, 90%, 95%, 99% - measures reliability of the estimator**

- **Confidence interval [L, U], where L and U both depend on the data (so that L and U are both random variables)**

functions of random sample

- **Reliability means if we repeat the experiments over and over again, 95% of times the interval will cover the true parameter**

Formal Statement

Let θ = unknown population parameter

Definition

Let $L < U$ be two numbers. If $[L, U]$ contain the parameter θ with probability $1-\alpha$. Then $[L, U]$ is called the two-sided confidence interval with confidence level $1-\alpha$.

$\alpha = 0.05$, e.g.,

Mathematically

$$P(\theta \in [L, U]) = 1 - \alpha$$

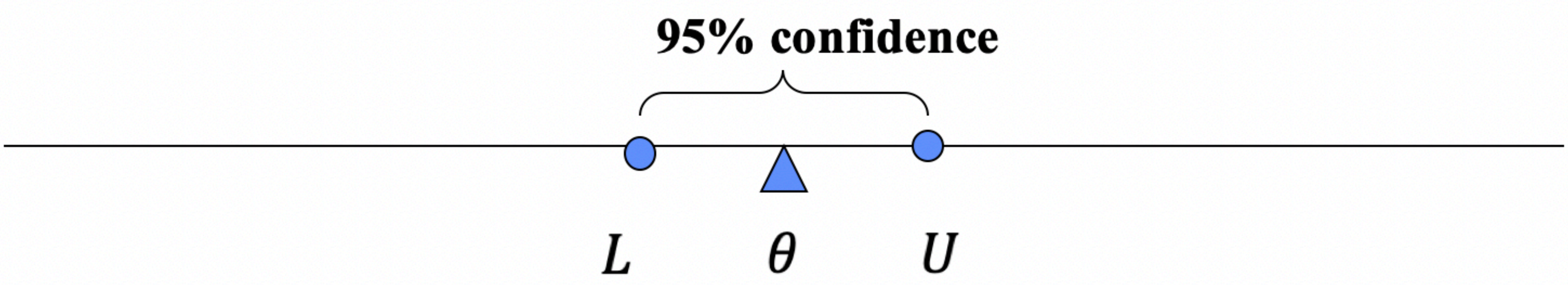
Important

The two sides $L < U$ are constructed from data, and thus random.
True parameter θ is fixed (deterministic) but unknown.

Interpretation

- For example, α = 0.05, we expect that 95% of all observed samples would give an interval that includes the true parameter.

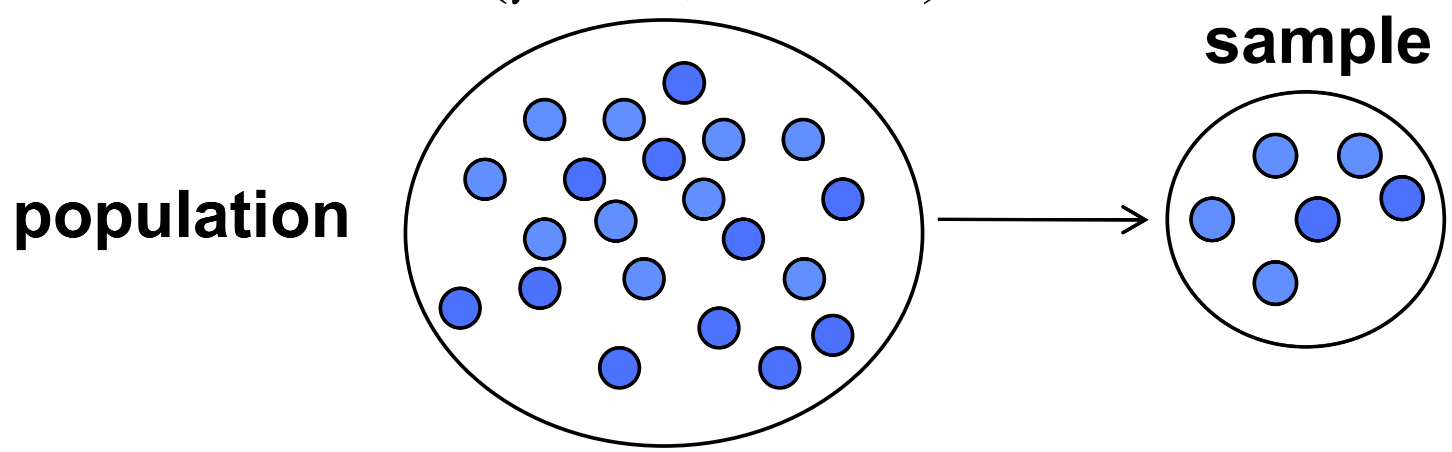
$$P(\theta \in [L, U]) = 1 - \alpha$$



Point Estimation Vs. Confidence Intervals

- The population distribution parameters are unknown. How to estimate them from samples?
  - Point Estimation / Interval Estimation

$$N(\mu = ?, \sigma^2 = ?)$$



$$X_1, X_2, \cdots, X_n$$

Point Estimation

$$\hat{\mu} = \overline{X}$$

$$\hat{\sigma}^2 = S^2$$

Interval Estimation

$$\Pr\{L \le \mu \le U\} = 1 - \alpha \qquad \Pr\{L' \le \sigma^2 \le U'\} = 1 - \alpha$$

$1 - \alpha$ **: Confidence level**

- Both lower and upper bounds are functions of a random sample

$$L = g(X_1, X_2, \cdots, X_n)$$

$$U = h(X_1, X_2, \cdots, X_n)$$

Key step: pivot quantity

How to find confidence interval?

Suppose we specify some $0 < \alpha < 1$ for $1 - \alpha$ confidence level. Now let's consider center this interval around a "pivot quantity".

For example, if we aim to estimate the mean μ. Let's choose $\bar{X}$ to be the pivot quantity. Then we want to find k such that

$$P(\bar{X} - k < \mu < \bar{X} + k) = 1 - \alpha$$

$$or$$

$$P(-k < \bar{X} - \mu < k) = 1 - \alpha$$

So, how to become a scientist?

- **Fortune Teller**

“I believe the answer is 75.6 meters”

- **Scientist**

“I believe the answer is 75.6 meters plus or minus 2.0”

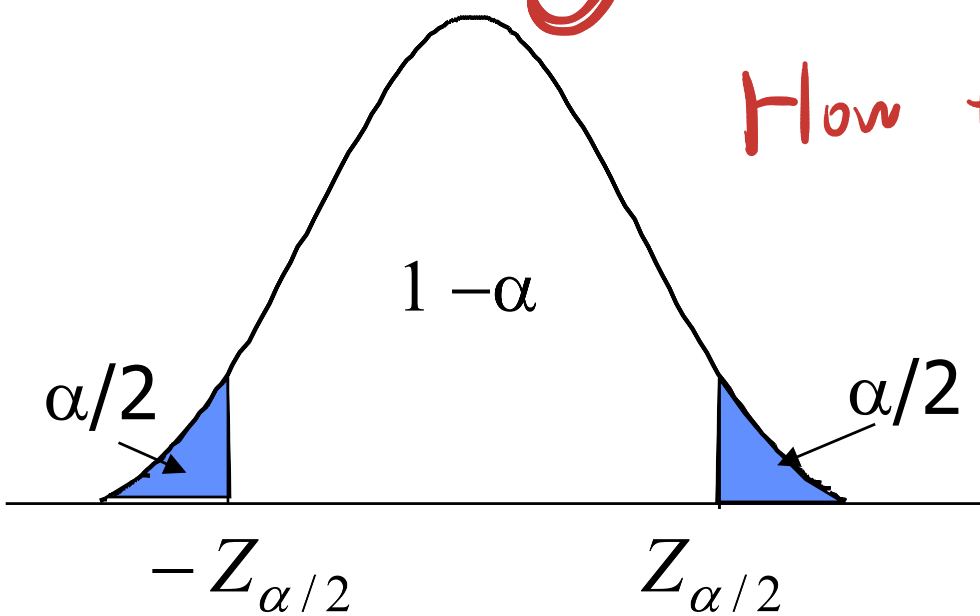
Two-sided Confidence Interval (CI) for Mean

100(1-α)% CI for Mean of a Normal Distribution (**two-sided**, **known variance**)

- **For mean μ from a normal population with known σ,**

$$X_1, \ldots, X_n \sim N(\mu, \sigma^2) \implies \overline{X} \sim N(\mu, \frac{\sigma^2}{n}) \implies \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

How to select k s.t. $P(\bar{x} - k \le \mu \le \bar{x} + k) = 1 - \alpha$?



$$P\left\{-z_{\alpha/2} \le \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right\} = 1 - \alpha$$

Confidence level

$\Rightarrow P\{-z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \le \bar{x} - \mu \le z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\} = 1 - \alpha$.

$\Leftrightarrow P\{\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\} = 1 - \alpha$

- population normal? $n > 30$
- CI lower, CI upper
- known variance

CI Lower **CI Upper**

$$\boxed{\bar{x} - Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}} \le \mu \le \boxed{\bar{x} + Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}}$$

This CI can be used for mean of **non-normal** distributions when n>30 (**CLT**)

Goal: $\mathcal{N}(\mu, \sigma^2)$. Using $X_1, \cdots, X_n$ to construct interval estimate of μ.

Step 1: Obtain point estimate.

$$\bar{X} \triangleq \frac{1}{n} \sum_{i=1}^{n} X_i$$

Step 2: distribution of point estimate.

$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Step 3: distribution of normalized estimate.

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$$

Step 4: Find $z_{\alpha/2}$ to ensure

$$P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \in [-z_{\alpha/2}, z_{\alpha/2}]\right) = 1-\alpha$$

holds.

$$\Rightarrow P\left(\bar{X}-\mu \in \left[-\frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right]\right) = 1-\alpha$$

$$\Rightarrow P\left(\mu \in \left[\bar{X}-\frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \bar{X}+\frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right]\right) = 1-\alpha$$

Example 1: z-interval

You want to rent an unfurnished 1Br apartment for the next semester. The mean monthly rent for a sample of 20 apartments in the local newspaper is $540 for a community where you want to move. Historically, the standard deviation of the rent is $80. Find a 90% confidence interval for the mean monthly rent in this community.

Example 1: z-interval

You want to rent an unfurnished 1Br apartment for the next semester. The **mean** monthly rent for a sample of ~~20~~ 40 apartments in the local newspaper is **$540** for a community where you want to move. **Historically**, the standard deviation of the rent is **$80**. Find a **90%** confidence interval for the mean monthly rent in this community.

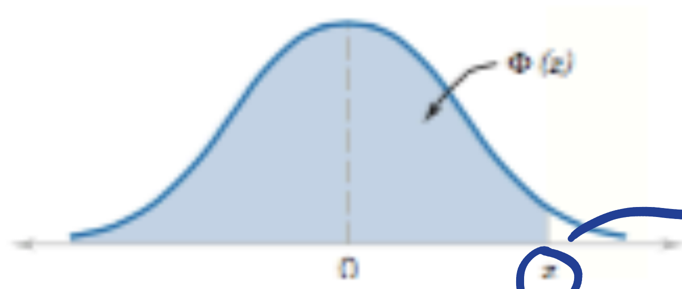
- $n = 40$
- One realization of $\overline{X}$ is 540
- $\sigma = 80$
- $\alpha = 0.1$ $\quad 1-\alpha = 90\%$
- CI estimate

$$\left[\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

$$= \left[540 - Z_{0.05}\cdot\frac{80}{\sqrt{40}},\ 540 + Z_{0.05}\frac{80}{\sqrt{40}}\right]$$

$$= \left[540 - 1.65\cdot\frac{80}{\sqrt{40}},\ 540 + 1.65\cdot\frac{80}{\sqrt{40}}\right]$$

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$



$Z_{0.65}$

0.05

$\Rightarrow Z_{0.05} = 1.65$

$z = 1.65$

Table III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427

Solution

Using CLT

Definition

If $\bar{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ CI on μ is given by

$$\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \bar{x} + z_{\alpha/2}\sigma/\sqrt{n} \quad (8\text{-}7)$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.

Large Sample Confidence Interval

EXAMPLE 8-4 Mercury Contamination

An article in the 1993 volume of the *Transactions of the American Fisheries Society* reports the results of a study to investigate the mercury contamination in largemouth bass. A sample of fish was selected from 53 Florida lakes, and mercury concentration in the muscle tissue, was measured (ppm). The mercury concentration values are

1.230	0.490	0.490	1.080	0.590	0.280	0.180	0.100	0.940
1.330	0.190	1.160	0.980	0.340	0.340	0.190	0.210	0.400
0.040	0.830	0.050	0.630	0.340	0.750	0.040	0.860	0.430
0.044	0.810	0.150	0.560	0.840	0.870	0.490	0.520	0.250
1.200	0.710	0.190	0.410	0.500	0.560	1.100	0.650	
0.270	0.500	0.770	0.730	0.340	0.170	0.160	0.270	

The summary statistics from Minitab are displayed below:

unknown!

$$\left[\bar{X} - z_{0.025}\frac{\sigma}{\sqrt{n}},\ \bar{X} + z_{0.025}\frac{\sigma}{\sqrt{n}}\right]$$

Descriptive Statistics: Concentration

Variable	N	Mean	Median	TrMean	StDev	SE Mean
Concentration	53	0.5250	0.4900	0.5094	0.3486	0.0479

Variable	Minimum	Maximum	Q1	Q3
Concentration	0.0400	1.3300	0.2300	0.7900

Figure 8-3(a) and (b) presents the histogram and normal probability plot of the mercury concentration data. Both plots indicate that the distribution of mercury concentration is not normal and is positively skewed. We want to find an approximate 95% CI on μ. Because $n > 40$, the assumption of normality is not necessary to use Equation 8-11. The required quantities are $n = 53$, $\bar{x} = 0.5250$, $s = 0.3486$, and $z_{0.025} = 1.96$. The approximate 95% CI on μ is

e.g., assume $n \geq 20$

$$\bar{x} - z_{0.025}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

$$0.5250 - 1.96\frac{0.3486}{\sqrt{53}} \leq \mu \leq 0.5250 + 1.96\frac{0.3486}{\sqrt{53}}$$

$$0.4311 \leq \mu \leq 0.6189$$

Practical Interpretation: This interval is fairly wide because there is a lot of variability in the mercury concentration measurements. A larger sample size would have produced a shorter interval.

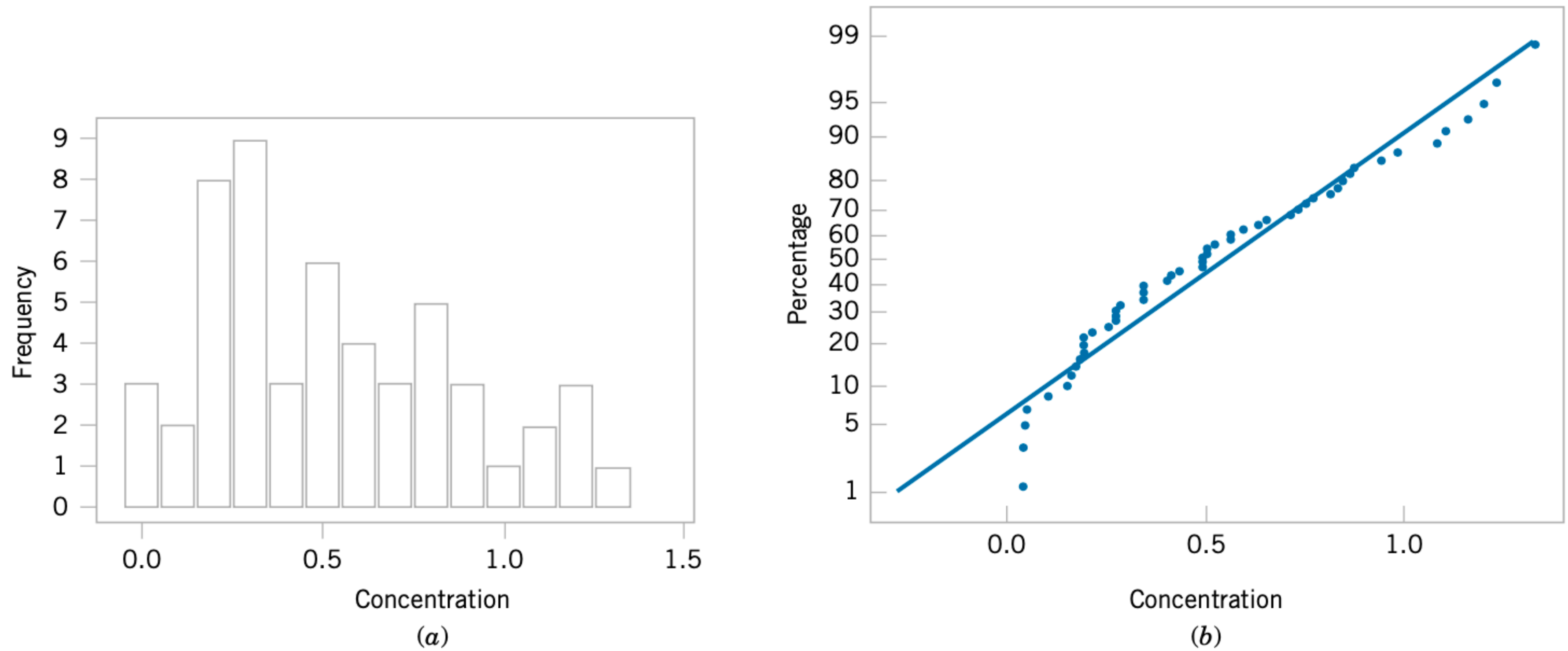
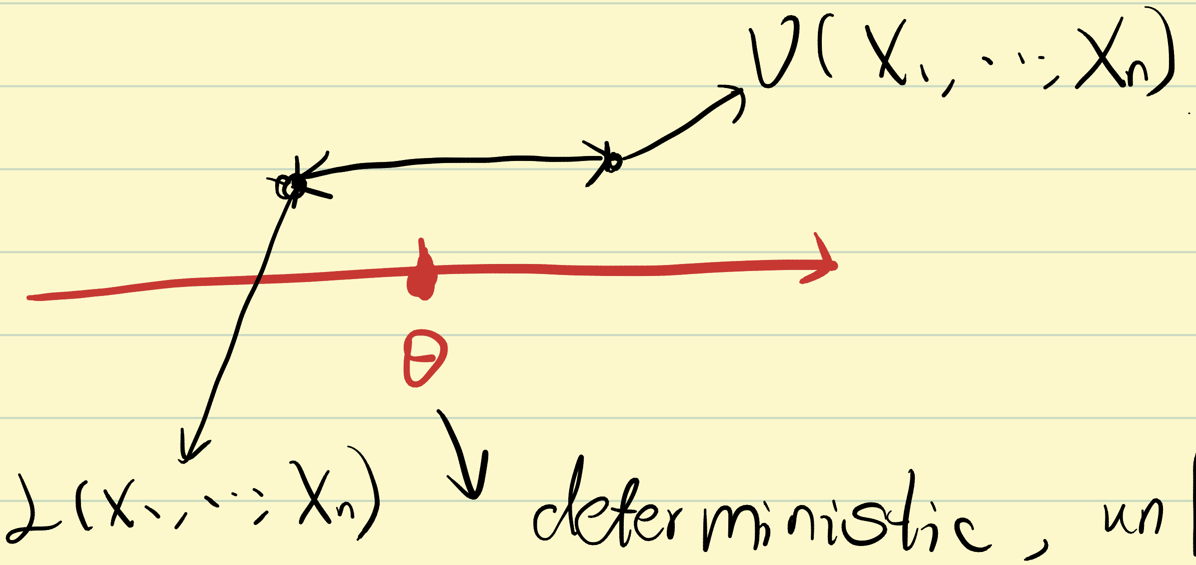


Figure 8-3 Mercury concentration in largemouth bass (a) Histogram. (b) Normal probability plot.

1. HW Question updated.

population for Q1, Q2 are normal

1. Confidence Interval



$\downarrow$ deterministic, unknown

$\boxed{X_1, \dots, X_n}$

$\downarrow$

$L(X_1, \dots, X_n)$

$U(X_1, \dots, X_n)$

---

2. For mean parameter θ, interval estimator

- point estimator: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Confidence Interval: $\left[\bar{X} - Z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}},\ \bar{X} + Z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}\right]$

If not, replace with S^2 when $n \geq 30$

1. σ is known 2. (population is normal) or (population is not normal, $n \geq 30$)

Choice of Sample Size

$$E = \text{error} = |\bar{x} - \mu|$$

$l = \bar{x} - z_{\alpha/2}\sigma/\sqrt{n}$ $\qquad \bar{x} \qquad \mu \qquad$ $u = \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}$

Deviation: $\left[\bar{x} - z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}\right]$ $\qquad \Delta = |\bar{x} - \mu|$

$[\bar{x} - \Delta,\ \bar{x} + \Delta]$

If $\bar{x}$ is used as an estimate of μ, we can be $100(1 - \alpha)\%$ confident that the error $|\bar{x} - \mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \tag{8-8}$$

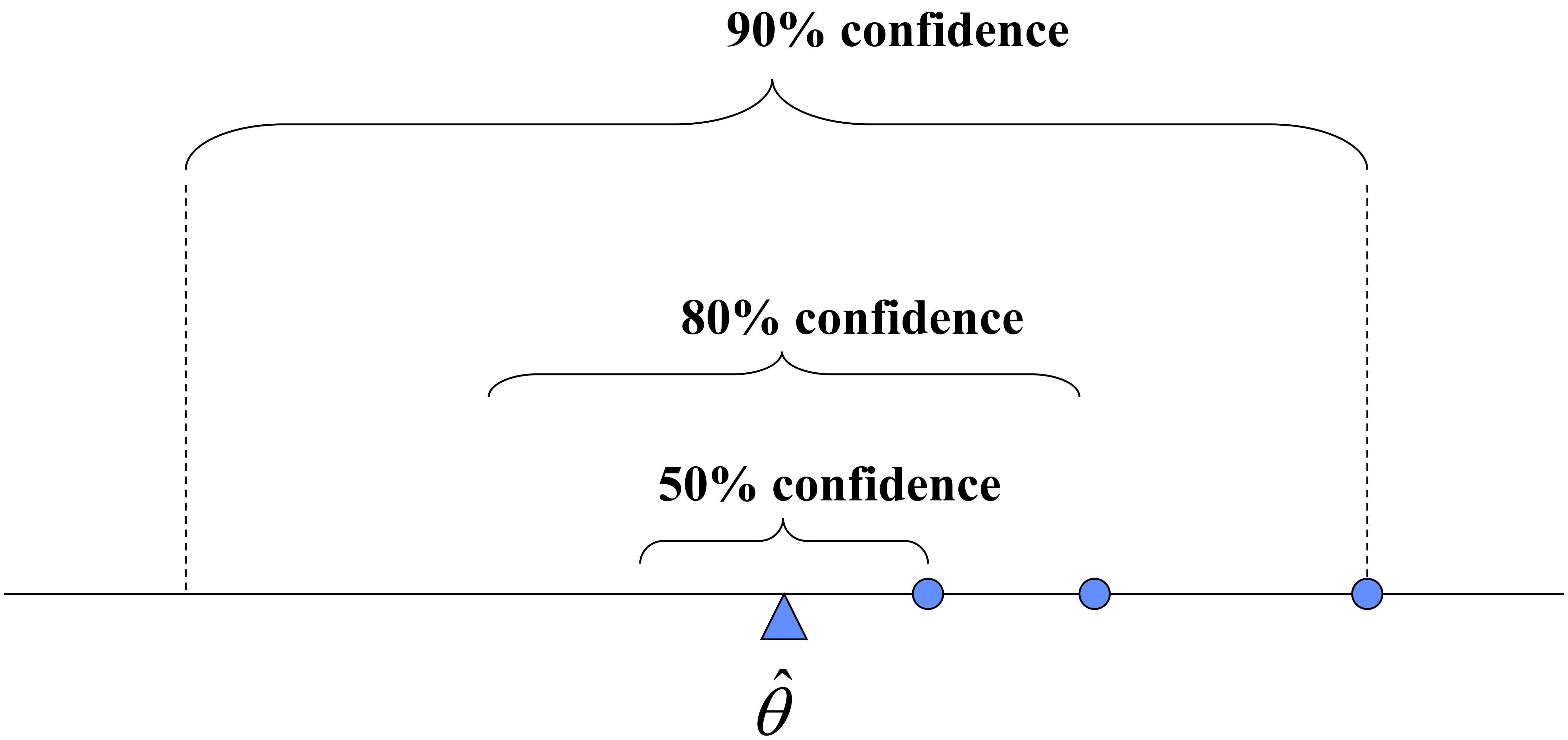
$\underbrace{E \geq z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}}$

$\sqrt{n}\cdot E \geq z_{\alpha/2}\cdot\sigma$

$\sqrt{n} \geq \frac{z_{\alpha/2}\cdot\sigma}{E}$

$\underbrace{n \geq \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2} \Leftrightarrow n = \left\lceil\left(\frac{z_{\alpha/2}\sigma}{E}\right)^2\right\rceil$

Constructing a Confidence Interval



The length of a confidence interval is a measure of the **precision** of estimation.

Choice of Sample Size

Example 8-2

$\left[\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right]$

$n = 16$

To illustrate the use of this procedure, consider the CVN test described in Example 8-1, and suppose that we wanted to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at 60°C has a length of at most 1.0*J*.

$\text{Length} = 2 \cdot z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \le 1$

$\Leftrightarrow \sqrt{n} \ge 2 z_{\alpha/2} \cdot \sigma$

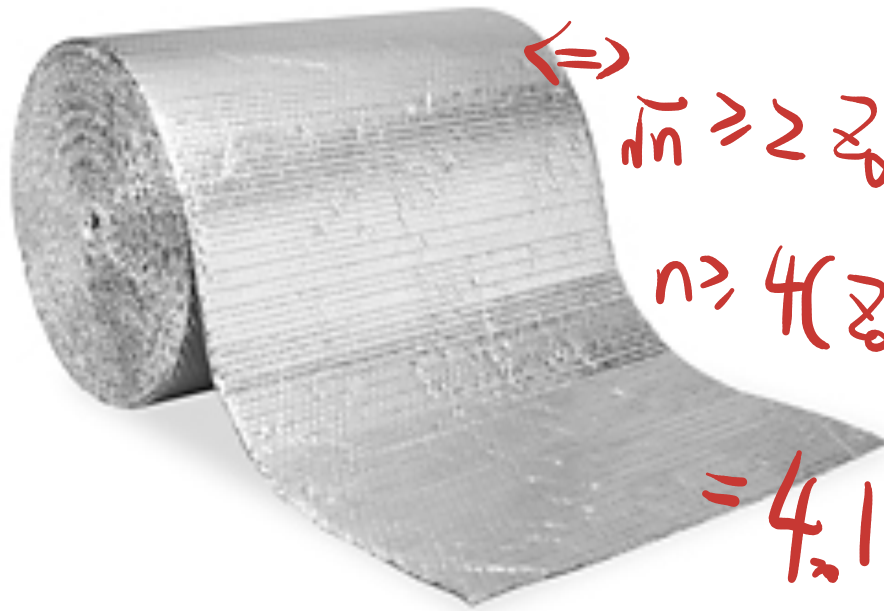
$n \ge 4 (z_{\alpha/2} \cdot \sigma)^2$

$= 4 \times 1.96^2$

$= 15.3$

EXAMPLE 8-1 Metallic Material Transition

ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (*J*) on specimens of A238 steel cut at 60°C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. Assume that impact energy is normally distributed with $\sigma = 1J$. We want to find a 95% CI for μ, the mean impact energy. The required quantities are $z_{\alpha/2} = z_{0.025} = 1.96$, $n = 10$, $\sigma = 1$, and $\bar{x} = 64.46$. The resulting ?



Solution

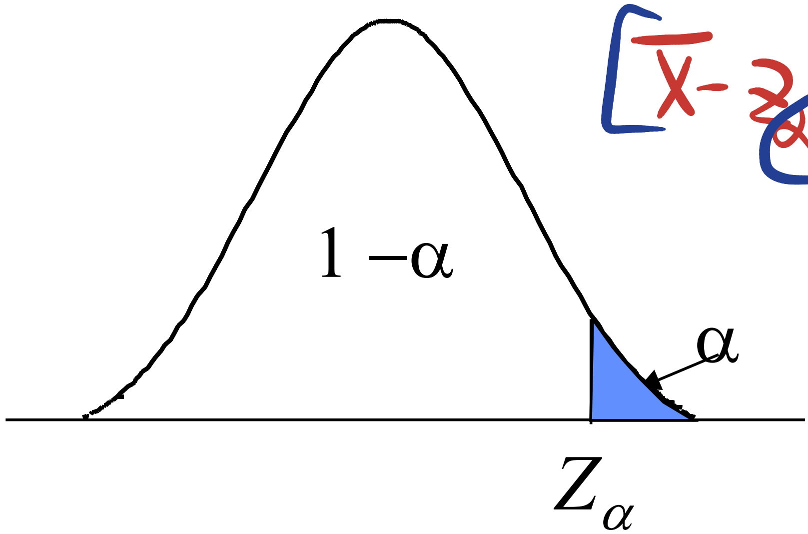
One-sided Confidence Interval

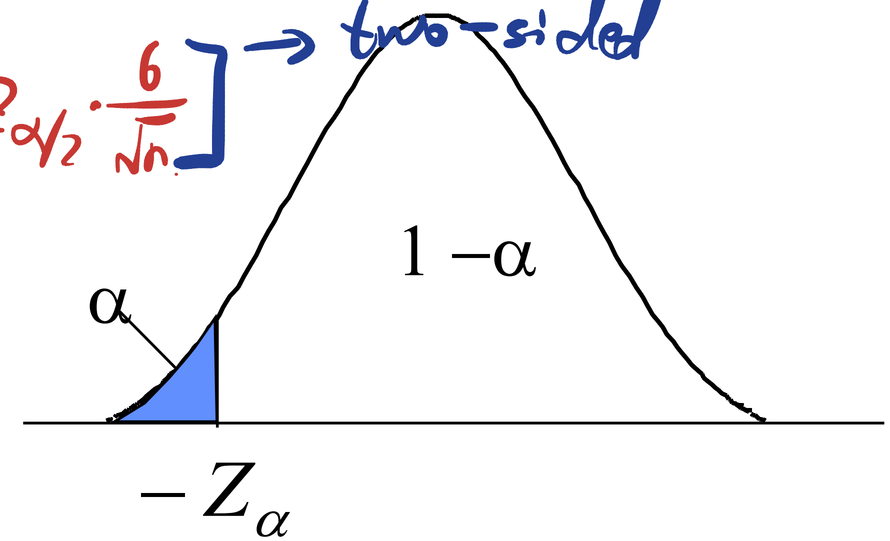
One-sided confidence interval

- Confidence interval in the forms of
  - $[-\infty, U]$ or
  - $[L, \infty]$
- We have some prior knowledge about the parameter: it has to be greater than or smaller than certain numbers

100(1-α)% CI for Mean of a Normal Distribution (one-sided, known variance)

$\left[\bar{X} - Z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}},\ \bar{X} + Z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}\right] \to$ two-sided





$$\Pr\left\{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le Z_\alpha\right\} = 1-\alpha$$

$$\mu \ge \bar{X} - Z_\alpha \times \frac{\sigma}{\sqrt{n}}$$

Lower bound

Two-sided: $Z_{\alpha/2}$

One-sided: Z_α

$$\Pr\left\{-Z_\alpha \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right\} = 1-\alpha$$

$$\mu \le \bar{X} + Z_\alpha \times \frac{\sigma}{\sqrt{n}}$$

Upper bound

This CI can be used for mean of non-normal distributions when n>30

Example

Example: The strength of a disposable plastic beverage container is being investigated. The strengths are normally distributed with a known standard deviation of 15 psi. A sample of 20 plastic containers has a mean strength of 246 psi. Compute the 95% lower bound CI for the process mean.

$$\left[\bar{X} - Z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}},\ \bar{X} + Z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}\right]$$

if I require CI

$$\mu \geq \underbrace{\bar{X} - Z_{\alpha}\cdot\frac{\sigma}{\sqrt{n}}}$$

$\bar{X} = 246$ $\quad \alpha = 0.05 \quad Z_{\alpha} = \text{qnorm}(1-\alpha)$

$\sigma = 15$ $\quad n = 20 \quad = 1.64$

$$\mu \geq 246 - 1.64\cdot\frac{15}{\sqrt{20}} = 240.50$$

100(1-α)% CI for Variance of a Normal Distribution

$N(\mu, \sigma^2)$ $X_1, \dots, X_n$ Goal: CI for parameter σ^2.

- Using chi-square, we can estimate a CI for the variance, σ^2, of a normal population.

• point estimator:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \mathrm{Var}(\text{data})$$

Two-sided CI

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow P\left\{\chi^2_{1-\alpha/2,n-1} \le \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{\alpha/2,n-1}\right\} = 1-\alpha \Rightarrow \boxed{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}$$

One-sided Upper bound

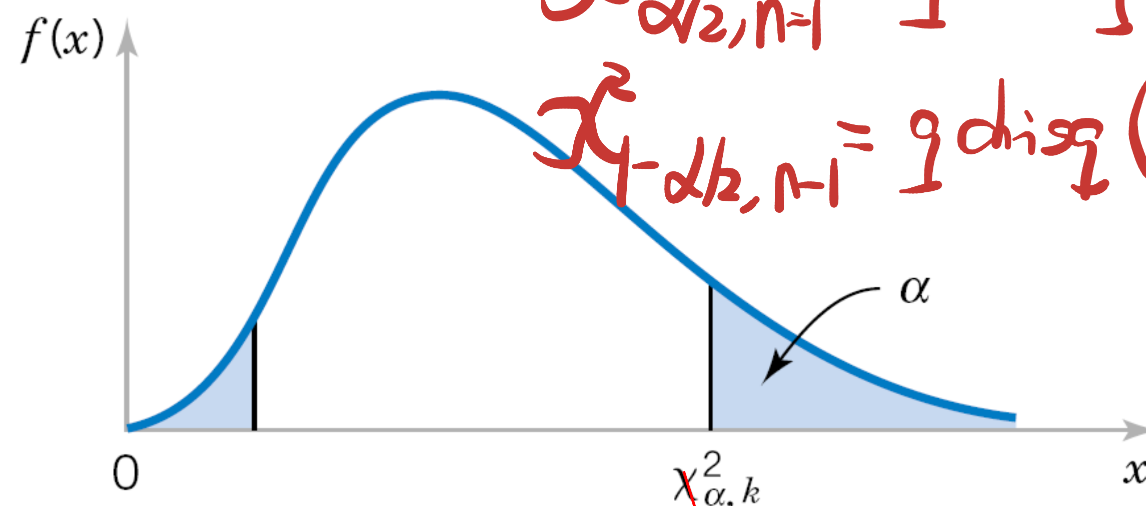
$$\boxed{\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}}}$$

$\chi^2_{\alpha/2,n-1} = \mathrm{qchisq}(1-\frac{\alpha}{2}, n-1)$

$\chi^2_{1-\alpha/2,n-1} = \mathrm{qchisq}(\frac{\alpha}{2}, n-1)$

One-sided Lower bound

$$\boxed{\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \le \sigma^2}$$



chi-squared critical Value

$$P\left(\chi^2_{1-\alpha/2,\,n-1} \le \frac{s^2(n-1)}{\sigma^2} \le \chi^2_{\alpha/2,\,n-1} \right) = 1-\alpha$$

$$\Longleftrightarrow \quad \sigma^2 \chi^2_{1-\alpha/2,\,n-1} \le s^2(n-1) \le \sigma^2 \chi^2_{\alpha/2,\,n-1}$$

$$\Longleftrightarrow \quad \frac{s^2(n-1)}{\chi^2_{\alpha/2,\,n-1}} \le \sigma^2 \le \frac{s^2(n-1)}{\chi^2_{1-\alpha/2,\,n-1}}$$

$$\Leftrightarrow \quad P\left(\sigma^2 \in \left[\frac{s^2(n-1)}{\chi^2_{\alpha/2,\,n-1}} ,\ \frac{s^2(n-1)}{\chi^2_{1-\alpha/2,\,n-1}} \right] \right) = 1-\alpha$$

EXAMPLE 8-6 Detergent Filling

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounce)^2. If the variance of fill volume is too large, an unacceptable proportion of bottles will be under- or overfilled. We will assume that the fill volume is approximately normally distributed. A 95% upper confidence bound is found from Equation 8-26 as follows:

$$\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{0.95,19}}$$

or

$$\sigma^2 \le \frac{(19)0.0153}{10.117} = 0.0287 \text{ (fluid ounce)}^2$$

This last expression may be converted into a confidence interval on the standard deviation σ by taking the square root of both sides, resulting in

$$\sigma \le 0.17$$

Practical Interpretation: Therefore, at the 95% level of confidence, the data indicate that the process standard deviation could be as large as 0.17 fluid ounce. The process engineer or manager now needs to determine if a standard deviation this large could lead to an operational problem with under-or over filled bottles.

$$\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}}$$

$$= \frac{19 \cdot s^2}{\chi^2_{0.95,19}}$$

$\alpha = 0.05$

$n = 20$

Two-sided interval

8-52. An article in the *Australian Journal of Agricultural Research* ["Non-Starch Polysaccharides and Broiler Performance on Diets Containing Soyabean Meal as the Sole Protein Concentrate" (1993, Vol. 44, No. 8, pp. 1483–1499)] determined that the essential amino acid (Lysine) composition level of soybean meals is as shown below (g/kg):

population is normal.

22.2	24.7	20.9	26.0	27.0
24.8	26.5	23.8	25.6	23.9

(a) Construct a 99% two-sided confidence interval for σ^2.

(b) Calculate a 99% lower confidence bound for σ^2.

(a) $$\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} , \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \right]$$

$n = 10$

data <- c(--, -, --, --)

$s^2 = Var(data) = 3.658$

$\alpha = 0.01$

$\chi^2_{0.005,9} = 23.59$ = qchisq(1-0.005, 9)

$\chi^2_{1-0.005,9} = 1.73$ = qchisq(0.005, 9)

(b) $$\sigma^2 \geq \frac{(n-1)s^2}{\chi^2_{\alpha,n-1}}$$

$n = 10$,

$s^2 = 3.658$

$s = 1.913$

99% C.I. for σ^2 : $1-\alpha = 0.99$
two-sided

$\alpha = 0.01$

$\alpha/2 = 0.005$

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,\,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\,n-1}}$$

$$\frac{9 \times 3.658}{23.59} \le \sigma^2 \le \frac{9 \times 3.658}{1.73}$$

$$1.396 \le \sigma^2 \le 19.030$$

$\chi^2_{0.005,9} = 23.59$, $\chi^2_{0.995,9} = 1.73$

one-sided 99% lower-CI. for σ^2:

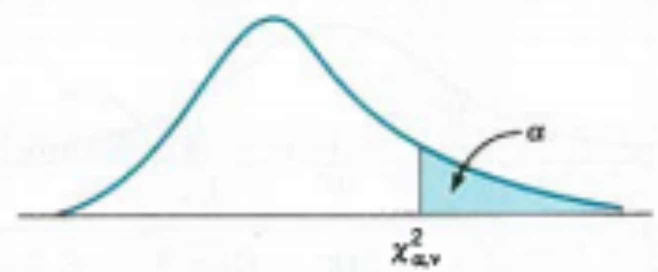
$$\sigma^2 \ge \frac{(n-1)s^2}{\chi^2_{\alpha,\,n-1}} = \frac{9 \times 3.658}{21.67} = 1.52$$

$\chi^2_{0.01,9} = 21.67$

Chi-square Table

■ **APPENDIX III**

Percentage Points of the χ^2 Distribution[a]



$\chi^2_{0.005,9} = 23.59$

ν	0.995	0.990	0.975	0.950	0.500	0.050	0.025	0.010	0.005
	α								
1	0.00 +	0.00 +	0.00 +	0.00 +	0.45	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	1.39	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	2.37	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	3.36	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	4.35	11.07	12.38	15.09	16.75
6	0.68	0.87	1.24	1.64	5.35	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	6.35	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	7.34	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	8.34	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	9.34	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	10.34	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	11.34	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	12.34	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	13.34	23.68	26.12	29.14	31.32
15	4.60	5.23	6.27	7.26	14.34	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	15.34	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	16.34	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	17.34	28.87	31.53	34.81	37.16
19	6.884	7.63	8.91	10.12	18.34	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	19.34	31.41	34.17	37.57	40.00
25	10.52	11.52	13.12	14.61	24.34	37.65	40.65	44.31	46.93

100(1-α)% CI for a Population Proportion with large sample size

$$X_1, X_2, \cdots, X_n \sim Bernoulli(p)$$

X $\quad P(X=0)=1-p \quad P(X=1)=p$

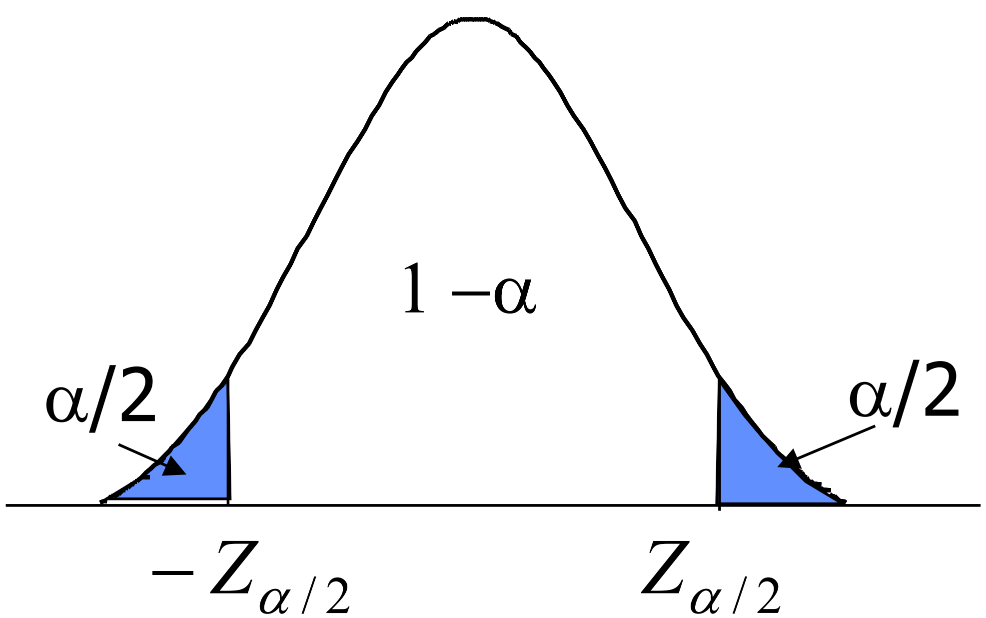
- Using CLT, we can estimate a CI for the population proportion.

- Goal: build CI for p
  - point estimator:

    $\hat{p} = \bar{X}$.

$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} \sim N(p, \frac{p(1-p)}{n}) \Rightarrow \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$$\Leftrightarrow Pr\left(Z_{\alpha/2} \le \frac{\hat{p}-p}{\sqrt{\frac{(1-p)p}{n}}} \le Z_{\alpha/2}\right) = 1-\alpha$$



$$\boxed{\hat{p} - Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

Two-sided CI

Example

EXAMPLE 8-7 Crankshaft Bearings

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Therefore, a point estimate of the proportion of bearings in the population that exceeds the roughness specification is $\hat{p} = x/n = 10/85 = 0.12$. A 95% two-sided confidence interval for p is computed from Equation 8-23 as

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

or

$$0.12 - 1.96\sqrt{\frac{0.12(0.88)}{85}} \le p \le 0.12 + 1.96\sqrt{\frac{0.12(0.88)}{85}}$$

which simplifies to

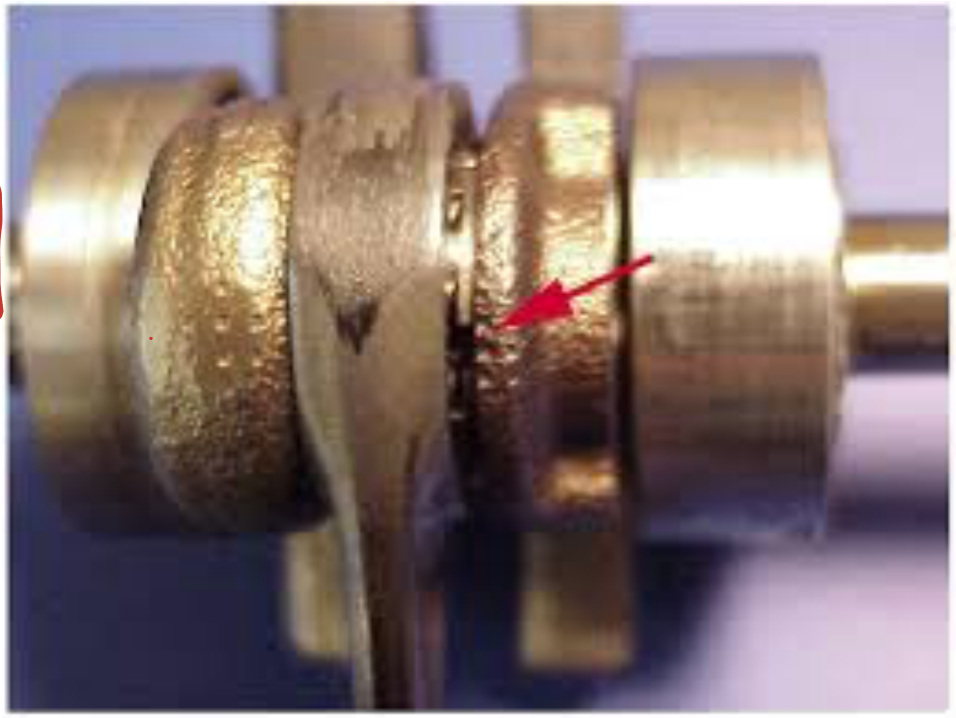
$$0.05 \le p \le 0.19$$

Practical Interpretation: This is a wide CI. While the sample size does not appear to be small ($n = 85$), the value of $\hat{p}$ is fairly small, which leads to a large standard error for $\hat{p}$ contributing to the wide CI.

$$\left[\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$\alpha = 0.05$ $\quad z_{\alpha/2} = 1.96$

$\hat{p} = 0.12$



Choice of Sample Size

The sample size for a specified value *E* is given by

$z_{\alpha/2} \cdot \sqrt{\frac{p(1-p)}{n}} \le E$ When p is known

$p(1-p) \le 0.25$

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1-p) \qquad (8\text{-}26)$$

An upper bound on *n* is given by (i.e., at least (1-α)100% confidence)

Assume p is unknown

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25) \qquad (8\text{-}27)$$

before

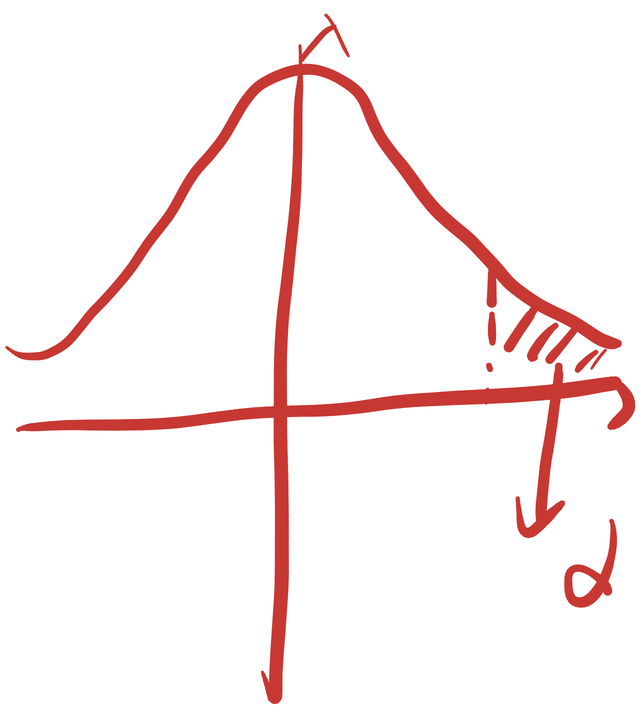
If $\bar{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error $|\bar{x}-\mu|$ will not exceed a specified amount E when the sample size is

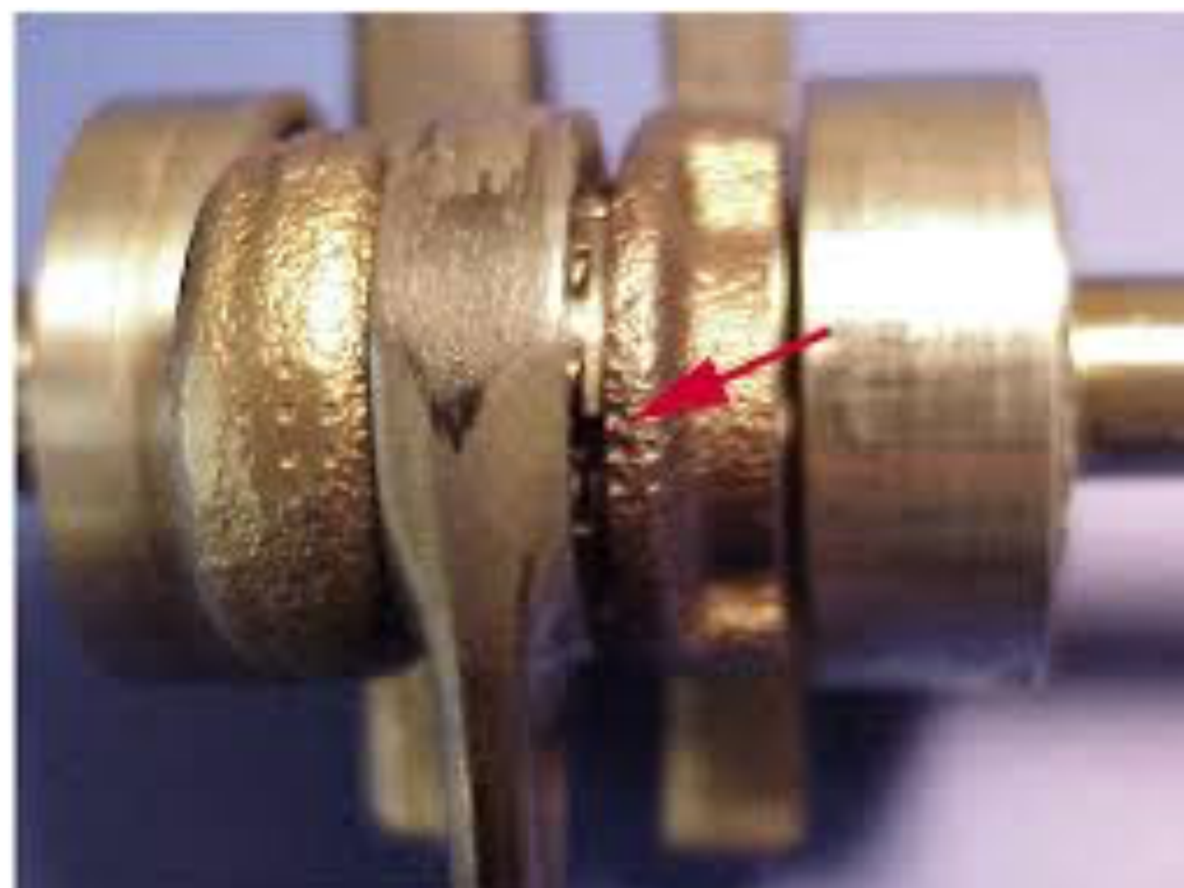
$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \qquad (8\text{-}8)$$

Example

Consider the situation in Example 8-7. How large a sample is required if we want to be 95% confident that the error in using $\hat{p}$ to estimate p is less than 0.05? Using $\hat{p} = 0.12$ as an initial estimate of p, we find from Equation 8-26 that the required sample size is

$$n = \left(\frac{z_{0.025}}{E}\right)^2 \hat{p}(1 - \hat{p}) = \left(\frac{1.96}{0.05}\right)^2 0.12(0.88) \cong 163$$





One-sided 100(1-α)% CI for a Population Proportion with large sample size

$$X_1, X_2, \cdots, X_n \sim Bernoulli(p)$$

- Using CLT, we can estimate a CI for the population proportion.

$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} \sim N(p, \frac{p(1-p)}{n})$$

$$\boxed{\hat{p} - Z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p}$$

One-sided Lower bound

$$\boxed{p \le \hat{p} + Z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

One-sided Upper bound

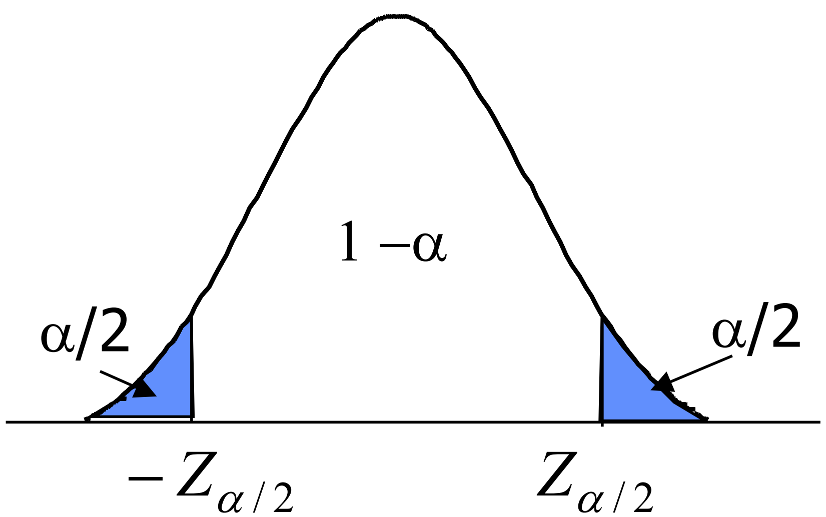
$$\boxed{\hat{p} - Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

Confidence Intervals for two-samples

100(1-α)% CI for Differences in Means of Two Normal Distributions (two-sided, known variances)

Two independent random samples from two Normal distributions with the known variances

$$\left.\begin{array}{l} X_1, X_2, \cdots, X_{n_1} \sim NID(\mu_1, \sigma_1^2) \\ \\ Y_1, Y_2, \cdots, Y_{n_2} \sim NID(\mu_2, \sigma_2^2) \end{array}\right\} \rightarrow \bar{X} - \bar{Y} \sim NID\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right) \rightarrow \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim NID(0,1)$$



$$\Pr\left\{ -Z_{\alpha/2} \le \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \le Z_{\alpha/2} \right\} = 1 - \alpha \longleftarrow$$

Confidence level

CI Lower **CI Upper**

$$(\bar{X} - \bar{Y}) - Z_{\alpha/2} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le (\bar{X} - \bar{Y}) + Z_{\alpha/2} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

This CI can be used for mean of non-normal distributions when n>30

Eg. Two-sided Confidence Interval

10-4. Two machines are used for filling plastic bottles with a net volume of 16.0 ounces. The fill volume can be assumed normal, with standard deviation $\sigma_1 = 0.020$ and $\sigma_2 = 0.025$ ounces. A member of the quality engineering staff suspects that both machines fill to the same mean net volume, whether or not this volume is 16.0 ounces. A random sample of 10 bottles is taken from the output of each machine.

Machine 1		**Machine 2**	
16.03	16.01	16.02	16.03
16.04	15.96	15.97	16.04
16.05	15.98	15.96	16.02
16.05	16.02	16.01	16.01
16.02	15.99	15.99	16.00

(b) Calculate a 95% confidence interval on the difference in means. Provide a practical interpretation of this interval.

defergent :

$n_1 = n_2 = 10$

$\bar{X} = 16.015$

$\bar{Y} = 16.005$

$\sigma_1 = 0.020$, $\sigma_2 = 0.025$, $1-\alpha = 0.95$,

two-sided C.I. for mean difference:

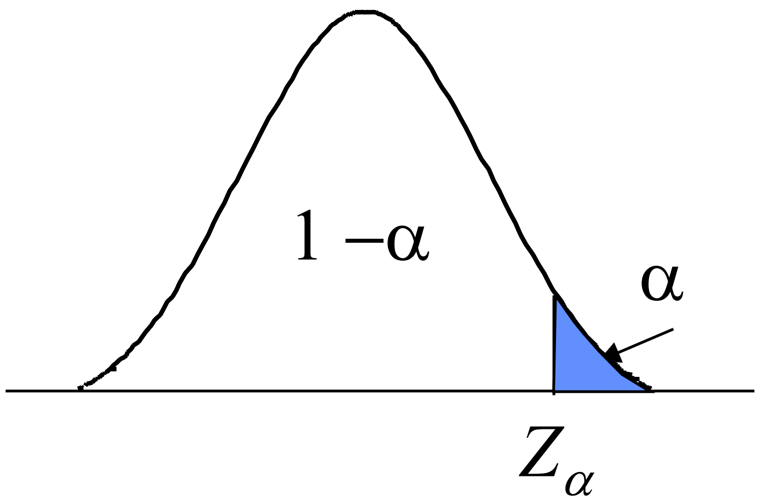
$$(\bar{X}-\bar{Y}) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \cdots$$

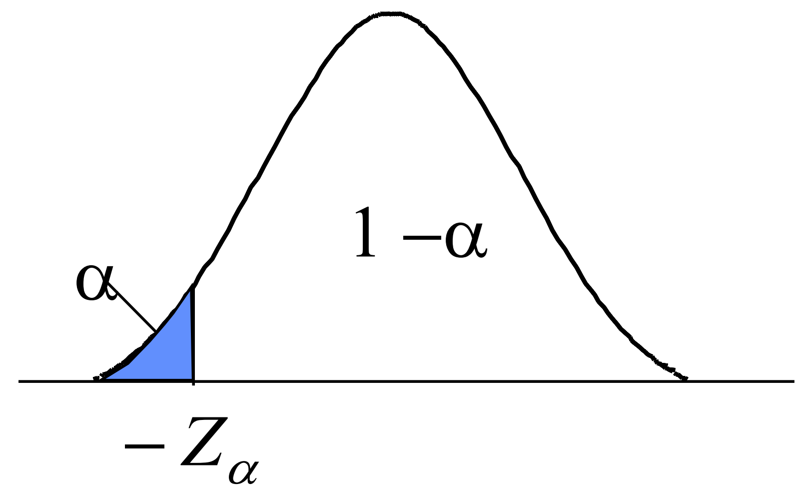
$\bar{X}-\bar{Y} = 0.01$, $z_{\alpha/2} = z_{0.025} = 1.96$

$$z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} = 1.96\times\sqrt{\frac{0.02^2}{10}+\frac{0.025^2}{10}} = 0.0198$$

$\Rightarrow$ 95% C.I. for $\mu_1 - \mu_2$ is $[-0.0098, 0.0298]$

100(1-α)% CI for Differences in Means of Two Normal Distributions (one-sided, **known** variances)





$$\Pr\left\{ \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}} \le Z_\alpha \right\} = 1-\alpha$$

$$\Pr\left\{ -Z_\alpha \le \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}} \right\} = 1-\alpha$$

$$\boxed{(\mu_1-\mu_2) \ge (\bar{X}-\bar{Y}) - Z_\alpha \times \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$$

Lower bound

$$\boxed{(\mu_1-\mu_2) \le (\bar{X}-\bar{Y}) + Z_\alpha \times \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$$

Upper bound

This CI can be used for mean of non-normal distributions when n>30

Summary

- Find interval estimate for parameters (rather than point estimator).
- Concepts: pivot quantity, confidence level, two-sided, one-side confidence interval

$$P(\theta \in [L, U]) = 1 - \alpha$$

- How to determine the minimum number of samples needed to achieve certain confidence width (i.e., uncertainty level).