Georgia Institute of Technology

ISyE3770 - Final Exam Practice

Instructor: Jie Wang

2024/04/30 (02:40PM - 05:30PM)

Name: ______________________________

GT Student ID: ______________________________

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1. This final exam practice contains 14 pages (including this cover page) and 9 questions. Total of points is 100.
2. In taking this examination, you are expected to adhere to the GT academic honor code. At a minimum, this requires that you utilize only the materials supplied to you and that you do not give help to, or accept help from, others.

Distribution of Marks

Question	Points	Score
1	8	
2	4	
3	4	
4	12	
5	6	
6	22	
7	12	
8	20	
9	12	
Total:	100	

Multiple Choice (12 points)
Remark: for each question, one and only one of four given choices (A, B, C, and D) is correct.

1. A textile fiber manufacturer is investigating a new drapery yarn, which the company claims has a mean thread elongation of 12 kilograms with a standard deviation of 0.5 kilograms. The company wishes to test the hypothesis $H_0 : \mu = 12$ against $H_1 : \mu < 12$, using the random sample of four specimens. Use sample mean $\bar{x}$ as the test statistic. Assume the critical region is defined as $\bar{x} < 11.5$ kilogram.

   (a) (4 points) Which one below corresponds to the Type-I error of the test:

   A. $\mathbb{P}\{\bar{X} < 11.5|\mu = 12\}$
   B. $\mathbb{P}\{\bar{X} < 11.5|\mu < 12\}$
   C. $\mathbb{P}\{\bar{X} > 11.5|\mu = 12\}$
   D. None above.

   (a) ____________

   (b) (4 points) Assuming the true mean is 11.25 kilograms. Which one below corresponds to the power of the test for this assumed parameter value:

   A. $\mathbb{P}\{\bar{X} < 11.5|\mu = 11.25\}$
   B. $\mathbb{P}\{\bar{X} < 11.5|\mu = 12\}$
   C. $\mathbb{P}\{\bar{X} > 11.5|\mu = 11.25\}$
   D. None above.

   (b) ____________

   Solution: According to the definition, type-I error refers to H_0 being true, but we reject it, and power refers to H_0 being false, and we successfully reject it. Therefore, we choose A for part (a) and A for part (b).

2. (4 points) Color blindness appears in the 1% of the people in a certain population. What is the lower bound on the number of observations from a random sample such that, the probability of containing at least one color-blinded person is 0.95?
   A. 299 B. 199 C. 399 D. 499

   2. ____________

   Solution: Let n be the number of observations. The probability that there exists at least one color-blinded person in a random sample of n observations should be:

   $$1 - \mathbb{P}\big(\text{No color-blinded person exists in a random sample of } n \text{ observations}\big)$$
   $$=1 - 0.99^n \geq 0.95.$$

In other words, we choose n such that $0.99^n \leq 0.05$, i.e., $n \geq \frac{\ln 0.05}{\ln 0.99} = 298.07$.
Hence, we choose A for Question 2.

3. (4 points) Which of the following **CANNOT** be used to describe the data variability?
A. Sample Range B. Sample Mean C. Sample Variance D. Sample IQR

3. ____________

Solution: Sample range, sample variance, and sample IQR can all be applied to describe data variability, while sample mean cannot. Therefore, we choose B for Question 3.

Regular Question (88 points)

4. Let X equal the weight in grams of a miniature candy bar. Assume that $\mu = \mathbb{E}[X] = 24.43$ and $\sigma^2 = \text{Var}(X) = 2.20$. Let $\overline{X}$ be the same mean of a random sample of $n = 40$ candy bars. Find
   (a) (4 points) $\mathbb{E}[\overline{X}]$
   (b) (4 points) $\text{Var}(\overline{X})$
   (c) (4 points) $\mathbb{P}\big(24.17 \leq \overline{X} \leq 24.82\big)$. approximately.

Solution: It is easy to see the mean and variance for the sample mean is:

$$\mathbb{E}[\overline{X}] = 24.43,$$
$$\text{Var}(\overline{X}) = \frac{2.20}{n} = 0.055.$$

Also, by central limit theorem, the sampling distribution for the sample mean is approximately $\mathcal{N}(24.43, 0.055)$. Therefore,

$$\begin{aligned}
&\mathbb{P}\big(24.17 \leq \overline{X} \leq 24.82\big) \\
&= \mathbb{P}(\overline{X} \leq 24.82) - \mathbb{P}(\overline{X} \leq 24.17) \\
&= \texttt{pnorm}(24.82, 24.43, \texttt{sqrt}(0.055)) - \texttt{pnorm}(24.17, 24.43, \texttt{sqrt}(0.055)) \\
&= 0.818.
\end{aligned}$$

5. Roll a fair four-sided die twice. Let X be the outcome on the first roll, and Y be the sum of the two rolls. Calculate
   (a) (1 point) $\mu_X = \mathbb{E}[X]$
   (b) (1 point) $\mu_Y = \mathbb{E}[Y]$
   (c) (1 point) $\sigma_X^2 = \mathrm{Var}(X)$
   (d) (1 point) $\sigma_Y^2 = \mathrm{Var}(Y)$
   (e) (1 point) $\mathrm{Cov}(X, Y)$
   (f) (1 point) $\rho(X, Y)$

Solution: According to the definition, X is uniformly distributed on $\{1, 2, 3, 4\}$. Hence,

$$\mathbb{E}[X] = \frac{1}{4}(1+2+3+4) = 2.5,$$
$$\mathbb{E}[X^2] = \frac{1}{4}(1^2+2^2+3^2+4^2) = 7.5,$$
$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1.25.$$

Also, Y equals $X + X'$, where X' is another random variable independent of X that is also uniformly distributed on $\{1, 2, 3, 4\}$. Hence,

$$\mathbb{E}[Y] = \mathbb{E}[X + X'] = 5,$$
$$\mathbb{E}[Y^2] = \mathbb{E}[(X + X')^2] = \mathbb{E}[X^2] + \mathbb{E}[(X')^2] + 2\mathbb{E}[X]\mathbb{E}[X'] = 27.5,$$
$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 2.5.$$

$$\begin{aligned}\mathrm{Cov}(X, Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X^2] + \mathbb{E}[X]\mathbb{E}[X'] - \mathbb{E}[X]\mathbb{E}[X] - \mathbb{E}[X]\mathbb{E}[X'] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1.25.\end{aligned}$$

Finally,

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \frac{\sqrt{2}}{2}./2051$$

6. We investigate the problem of whether people gain weight as they age or not using data. We have $n = 250$ observations on the age (x) and weight (y). The linear regression output is attached below:

```
> model <- lm(weight~age)
> summary(model)

Call:
lm(formula = weight ~ age)

Residuals:
    Min      1Q  Median      3Q     Max
-70.538 -14.853   0.268  17.469  65.191

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 178.46936    3.41128   52.32   <2e-16 ***
age           0.02196    0.08440    0.26    0.795
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 27.03 on 248 degrees of freedom
Multiple R-squared:  0.000273,  Adjusted R-squared:  -0.003758
F-statistic: 0.06771 on 1 and 248 DF,  p-value: 0.7949
```

(a) (4 points) Write down the regression model found.

(b) (4 points) Estimate σ^2.

(c) (4 points) What percentage of the variability in the data does this model account for?

(d) (3 points) Use the equation of the fitted line to predict the weight that would be observed, on average, for a man who is 25 years old, and for a man who is 65 years old, respectively.

(e) (3 points) Suppose that the observed weight of a 25-year-old man is 170lb, and the observed weight of a 65-year-old man is 180lb. Find the residual for these two observations.

(f) (4 points) What is the interpretation of the observation that the value of Pr$(>|t|)$ for the age variable is much larger than 0?

Solution to Problem 6:

Solution:

(a) $\hat{y} = 178.47 + 0.02x$.

(b) $\hat{\sigma}^2 = 27.03^2 = 730.62$.

(c) This model accounts for 0.027% of the data variability.

(d) On average, a man who is 25 years old has weight 178.97lb, and a man who is 65 years old has weight 179.77lb.

(e) The residual for the first observation is

$$\text{Residual} = y - \hat{y} = 170 - 178.97 = -8.97,$$

and the residual for the second observation is

$$\text{Residual} = y - \hat{y} = 180 - 179.77 = 0.23.$$

(f) For the age variable, we can see the value of `Pr`$(> |t|)$ is 0.795, which represents the p-value of the hypothesis testing $H_0: \ \beta_1 = 0, H_1: \ \beta_1 \neq 0$, where β_1 denotes the slope. Therefore, this indicates there is no strong evidence to claim that the age variable has a significant impact on the regression model.

7. (12 points) A random variable has probability density function

$$f(x;\theta) = \frac{1}{\theta^2} x^{\frac{1-\theta^2}{\theta^2}}, \qquad 0 < x < 1, 0 < \theta < \infty.$$

Now, given samples $X_1, \ldots, X_n$, derive the maximum likelihood estimator for the parameter θ.

Solution: We first write down and simplify the log-likelihood function:

$$\begin{aligned} l(\theta; X_1, \ldots, X_n) &= \sum_{i=1}^{n} \log\left(\frac{1}{\theta^2} X_i^{\frac{1-\theta^2}{\theta^2}}\right) \\ &= \sum_{i=1}^{n} \left[-2\log\theta + \frac{1-\theta^2}{\theta^2}\log X_i\right] \\ &= -2n\log\theta + \frac{1-\theta^2}{\theta^2}\sum_{i=1}^{n}\log X_i. \end{aligned}$$

Taking the derivative and set it equal to 0, we obtain

$$-\frac{2n}{\widehat{\theta}_{\text{MLE}}} - \frac{2}{\widehat{\theta}^3_{\text{MLE}}}\sum_{i=1}^{n}\log X_i = 0.$$

Therefore, we obtain

$$\widehat{\theta}_{\text{MLE}} = \sqrt{-\frac{1}{n}\sum_{i=1}^{n}\log X_i}.$$

8. Medical researchers have developed a new artificial heart constructed primarily of titanium and plastic. The heart will last and operate almost indefinitely once it is implanted in the patient's body, but the battery pack needs to be recharged about every four hours. A random sample of 10 battery packs is selected and subject to a life test. The results of these 10 tests are as follows:

$$4.37, \quad 4.19, \quad 3.93, \quad 4.21, \quad 4.11, \quad 3.84, \quad 4.14, \quad 4.01, \quad 4.16, \quad 4.12.$$

Assume that battery life is normally distributed with standard deviation $\sigma = 0.2$ hour.

(a) (10 points) Is there evidence to support the claim that mean battery life exceeds 4 hours? Use significance level $\alpha = 0.05$.

(b) (10 points) Compute the power of the test if the true mean battery life is 4.2 hours.

Solution:

(a)
- Hypothesis testing: $H_0: \ \mu = 4$, $H_1: \ \mu > 4$, where $\mu_0 = 4$.
- Testing statistics: $Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$.
- We reject H_0 if $Z_0 > Z_\alpha$, where $\alpha = 0.05$.
- After calculation, $Z_0 = 1.707$ and $Z_\alpha = 1.644$.
- Therefore, we claim that mean battery life exceeds 4 hours with significance level $\alpha = 0.05$.

(b) The power equals the probability of rejecting H_0 provided that H_1 is true. In such case,

$$\begin{aligned} \text{Power} &= \mathbb{P}(Z_0 > 1.644 \mid \mu = 4.2) \\ &= \mathbb{P}(\mathcal{N}(0.2 \cdot \sqrt{n}/\sigma, 1) > 1.644) = 0.936. \end{aligned}$$

9. (12 points) To study the pH of rain in Hall County, Georgia, we collect 39 samples:

5.37, 5.47, 5.38, 4.63, 5.37, 3.74, 3.71, 4.96, 4.64, 5.11, 5.65, 5.55, 4.00, 5.62, 4.57, 4.64, 5.48, 4.60, 4.54, 4.51, 4.86, 4.56, 4.61, 4.32, 3.98, 5.70, 4.15, 3.98, 5.65, 3.11, 5.03, 4.62, 4.50, 4.35, 4.16, 4.64, 5.12, 3.71, 4.64.

The sample mean of the pH value is 4.698, and the sample variance of the pH value is 0.3963. Find the two-sided 95% confidence interval for the variance of pH.

Solution: In this case, the parameters

$$n = 39, \qquad \bar{X} = 4.698, \qquad S^2 = 0.6295, \qquad \alpha = 0.05.$$

We construct the confidence interval as

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}.$$

Next, we substitute the values of those parameters and compute the critical values $\chi^2_{\alpha/2,n-1} = \chi^2_{0.025,38} = 56.90$ and $\chi^2_{1-\alpha/2,n-1} = \chi^2_{0.975,38} = 22.88$ to obtain

$$0.2646 \leq \sigma^2 \leq 0.6582.$$

Discrete Distributions

Bernoulli $0 < p < 1$	$f(x) = p^x(1-p)^{1-x}, \qquad x = 0, 1$ $M(t) = 1 - p + pe^t, \qquad -\infty < t < \infty$ $\mu = p, \qquad \sigma^2 = p(1-p)$
Binomial $b(n, p)$ $0 < p < 1$	$f(x) = \dfrac{n!}{x!(n-x)!} p^x(1-p)^{n-x}, \qquad x = 0, 1, 2, \ldots, n$ $M(t) = (1 - p + pe^t)^n, \qquad -\infty < t < \infty$ $\mu = np, \qquad \sigma^2 = np(1-p)$
Geometric $0 < p < 1$	$f(x) = (1-p)^{x-1}p, \qquad x = 1, 2, 3, \ldots$ $M(t) = \dfrac{pe^t}{1-(1-p)e^t}, \qquad t < -\ln(1-p)$ $\mu = \dfrac{1}{p}, \qquad \sigma^2 = \dfrac{1-p}{p^2}$
Negative Binomial $0 < p < 1$ $r = 1, 2, 3, \ldots$	$f(x) = \dbinom{x-1}{r-1} p^r(1-p)^{x-r}, \qquad x = r, r+1, r+2, \ldots$ $M(t) = \dfrac{(pe^t)^r}{[1-(1-p)e^t]^r}, \qquad t < -\ln(1-p)$ $\mu = r\left(\dfrac{1}{p}\right), \qquad \sigma^2 = \dfrac{r(1-p)}{p^2}$
Poisson $\lambda > 0$	$f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots$ $M(t) = e^{\lambda(e^t - 1)}, \qquad -\infty < t < \infty$ $\mu = \lambda, \qquad \sigma^2 = \lambda$
Uniform $m > 0$	$f(x) = \dfrac{1}{m}, \qquad x = 1, 2, \ldots, m$ $\mu = \dfrac{m+1}{2}, \qquad \sigma^2 = \dfrac{m^2-1}{12}$

Continuous Distributions

Beta
$\alpha > 0$
$\beta > 0$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1$$

$$\mu = \frac{\alpha}{\alpha+\beta}, \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

Chi-square
$\chi^2(r)$
$r = 1, 2, \ldots$

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \qquad 0 < x < \infty$$

$$M(t) = \frac{1}{(1-2t)^{r/2}}, \qquad t < \frac{1}{2}$$

$$\mu = r, \qquad \sigma^2 = 2r$$

Exponential
$\theta > 0$

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$M(t) = \frac{1}{1-\theta t}, \qquad t < \frac{1}{\theta}$$

$$\mu = \theta, \qquad \sigma^2 = \theta^2$$

Gamma
$\alpha > 0$
$\theta > 0$

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad 0 < x < \infty$$

$$M(t) = \frac{1}{(1-\theta t)^\alpha}, \qquad t < \frac{1}{\theta}$$

$$\mu = \alpha\theta, \qquad \sigma^2 = \alpha\theta^2$$

Normal
$N(\mu, \sigma^2)$
$-\infty < \mu < \infty$
$\sigma > 0$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$$

$$M(t) = e^{\mu t + \sigma^2 t^2/2}, \qquad -\infty < t < \infty$$

$$E(X) = \mu, \qquad \mathrm{Var}(X) = \sigma^2$$

Uniform
$U(a, b)$
$-\infty < a < b < \infty$

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

$$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \qquad t \ne 0; \qquad M(0) = 1$$

$$\mu = \frac{a+b}{2}, \qquad \sigma^2 = \frac{(b-a)^2}{12}$$

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

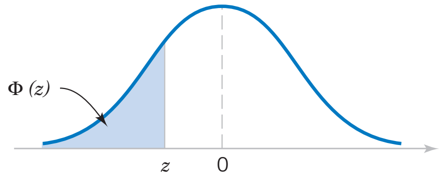

Table III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

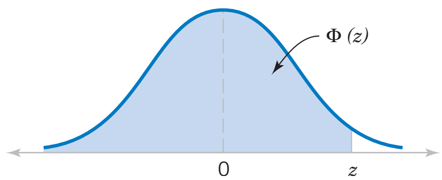

Table III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

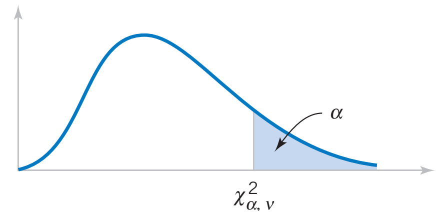

Table IV Percentage Points $\chi^2_{\alpha,\nu}$ of the Chi-Squared Distribution

ν \ α	.995	.990	.975	.950	.900	.500	.100	.050	.025	.010	.005
1	.00+	.00+	.00+	.00+	.02	.45	2.71	3.84	5.02	6.63	7.88
2	.01	.02	.05	.10	.21	1.39	4.61	5.99	7.38	9.21	10.60
3	.07	.11	.22	.35	.58	2.37	6.25	7.81	9.35	11.34	12.84
4	.21	.30	.48	.71	1.06	3.36	7.78	9.49	11.14	13.28	14.86
5	.41	.55	.83	1.15	1.61	4.35	9.24	11.07	12.83	15.09	16.75
6	.68	.87	1.24	1.64	2.20	5.35	10.65	12.59	14.45	16.81	18.55
7	.99	1.24	1.69	2.17	2.83	6.35	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	7.34	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	8.34	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	9.34	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	10.34	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	11.34	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	12.34	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	13.34	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.27	7.26	8.55	14.34	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	15.34	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	16.34	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.87	17.34	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	24.34	34.28	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.65
28	12.46	13.57	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	59.33	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	69.33	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	79.33	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	89.33	107.57	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	99.33	118.50	124.34	129.56	135.81	140.17

ν = degrees of freedom.