Section 3.2 exponential, gamma, chi-Square Distributions

Definition 3.2-5 [chi-square distribution]

Let X have a Gamma distribution with $\theta = 2, \alpha = \frac{r}{2}$, r is a integer.

The pdf of X is $f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad x > 0.$

Then X has **chi-square distribution** with ***r*** degrees of freedom, which is denoted by $X \sim \chi^2(r)$.

➢Mean and Variance

$$E(X) = \alpha\theta = \frac{r}{2} \cdot 2 = r, \qquad Var(X) = \alpha\theta^2 = \frac{r}{2} \cdot 2^2 = 2r.$$

Just change the α and θ in mean and variance of **Gamma distribution** into it.

$$mgf: M(t) = (\frac{1}{1-\theta t})^{\alpha} = (1-2t)^{-r/2}, \quad t < \frac{1}{2}$$

Remark: chi-square distribution plays an important role in Statistics, the tables of the values for *cdf* of chi-square distribution are given in our textbook!

$$F(x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} dw.$$

for selected values of r and x. (You can check Table IV in Appendix B in textbook.)

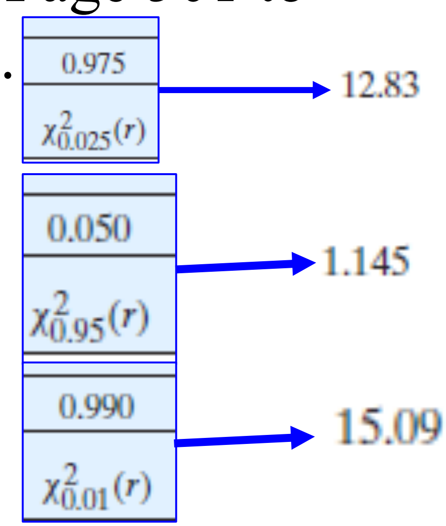
Example 2

Let X have a chi-square distribution with r=5 degrees of freedom. Then using table IV in Appendix B on Page 501 to find $P(1.145 \le X \le 12.83)$ and $P(X > 15.09)$.

Solution :

$$P(1.145 \le X \le 12.83) = F(12.83) - F(1.145)$$
$$= (1 - 0.025) - (1 - 0.95)$$
$$= 0.925$$

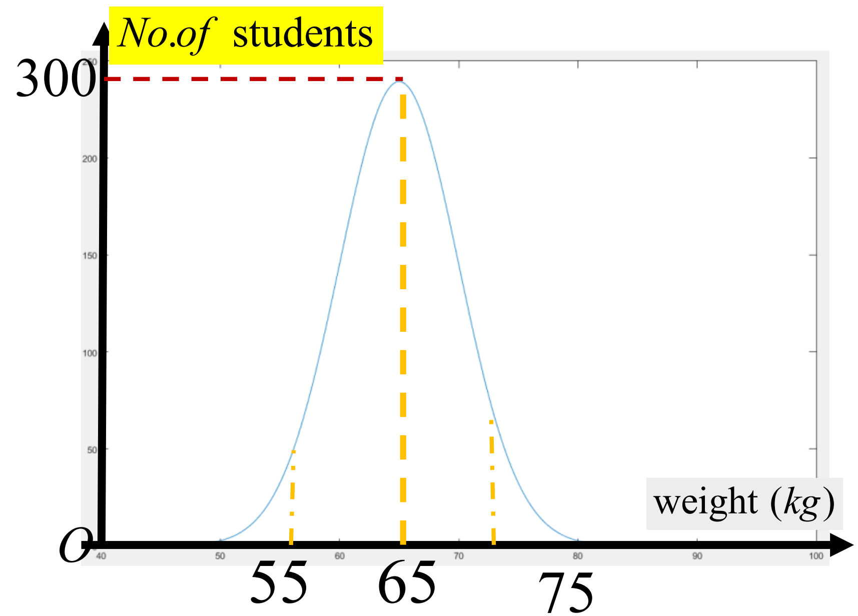
$$P(X > 15.09) = 1 - F(15.09) = 1 - (1 - 0.99) = 0.01$$



Section 3.3 Normal distribution

Situation: When observed over a large population, many variables have a “bell-shaped” relative frequency distribution.

- Weight of male students in CUHK(sz)
- Height
- TOFEL,IELTS test score



A very useful family of probability distributions for such variables are the normal distributions.

Definition 3.3-1 [Normal distribution]

A continuous *RV X* is said to be ***normal*** or ***Gaussian*** if has a pdf of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2} \right] \qquad -\infty < x < +\infty, \ \mu \ and \ \sigma^2 \ are \ real.$$

where μ *and* σ^2 are two parameters characterizing the normal distribution.

Briefly, $X \sim N(\mu, \sigma^2)$.

What's interpretation of μ *and* σ^2 ? consider mean and Variance.

- *f*(*x*) is a well-defined *pdf*

① $f(x) \geq 0$ for all x.

② We have to check whether $\int_{-\infty}^{+\infty} f(x)dx = 1$.

We set $I = \int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2})dx.$

By change of variable, we let $z = \frac{x-\mu}{\sigma}$.

$\Rightarrow I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})dz.$ Since $I > 0$, we only need to show $I^2 = 1$.

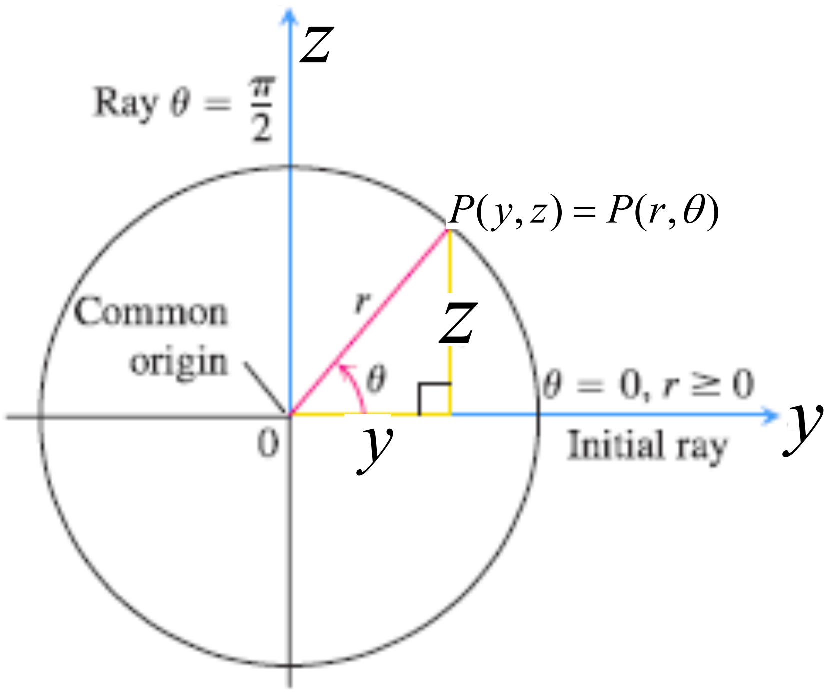
➢ $f(x)$ is a well-defined *pdf* (c.n.t.)

$$I^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}dz\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}}dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}e^{-\frac{y^2}{2}}dzdy = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{y^2+z^2}{2}}dydz.$$

coordinate change: $\begin{cases} y = r\cos\theta \\ z = r\sin\theta \end{cases}$ (polar coordinate)

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} e^{-\frac{r^2}{2}}\cdot rdrd\theta = \frac{1}{2\pi}\int_0^{2\pi}d\theta\int_0^{+\infty} e^{-\frac{r^2}{2}}\cdot rdr = \frac{1}{2\pi}\int_0^{2\pi}d\theta\int_0^{+\infty} e^{-\frac{r^2}{2}}d\frac{r^2}{2}$$

$$= \frac{1}{2\pi}\times 2\pi\times\left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} = [0-(-1)] = 1.$$



Thus, $I = 1$, and we have shown that $f(x)$ has the properties of a pdf.

If you don't know some specific steps to derive this conclusion, memory is a good solution.

➢ Mean and Variance (idea of mgf)

Assume $X \sim N(\mu, \sigma^2)$. $\quad M(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\cdot\frac{(x-\mu)^2}{\sigma^2}} dx.$

$$e^{tx} e^{-\frac{1}{2}\cdot\frac{(x-\mu)^2}{\sigma^2}} = \exp\left\{-\frac{1}{2\sigma^2}\left[x^2 - 2(\mu+\sigma^2 t)x + \mu^2\right]\right\}$$

consider $\quad x^2 - 2(\mu+\sigma^2 t)x + \mu^2 = \left[x - (\mu+\sigma^2 t)\right]^2 - 2\mu\sigma^2 t - \sigma^4 t^2,$

$$M(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left[x - (\mu+\sigma^2 t)\right]^2\right\} dx \cdot \exp\left(\frac{-2\mu\sigma^2 t - \sigma^4 t^2}{-2\sigma^2}\right)$$

Recall that $I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx = 1,$ independent of μ.

Therefore, by changing μ into $\mu + \sigma^2 t,$ we have:

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left[x - (\mu+\sigma^2 t)\right]^2\right\} dx = 1$$

$$\Rightarrow M(t) = \exp\left(\frac{-2\mu\sigma^2 t - \sigma^4 t^2}{-2\sigma^2}\right) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2).$$

M(0)=1

How to derive the mean and Variance based on *mgf*?

➢ Mean and Variance (c.n.t.)

$$M'(t) = (\mu + \sigma^2 t)\exp(\mu t + \frac{1}{2}\sigma^2 t^2) \qquad \Rightarrow M'(0) = \mu.$$

$$M''(t) = \sigma^2 \exp(\mu t + \frac{1}{2}\sigma^2 t^2) + (\mu + \sigma^2 t)^2 \exp(\mu t + \frac{1}{2}\sigma^2 t^2) \qquad \Rightarrow M''(0) = \mu + \sigma^2.$$

Recall that E(X)=$M'(0) = \mu, \quad Var(X) = E(X^2) - [E(X)]^2 = \sigma^2.$

For $X \sim N(\mu, \sigma^2),\ E(X) = \mu,\ Var(X) = \sigma^2.$

Example 1 (Page 115)

A $RV\ X$ has its pdf

$$f(x) = \frac{1}{\sqrt{32\pi}} \exp\left[-\frac{(x+7)^2}{32}\right], \qquad -\infty < x < +\infty.$$

compute the mgf of X.

Solution :

Obviously, $X \sim N(-7, 16) \Rightarrow E(X) = -7,\ Var(X) = 16.$

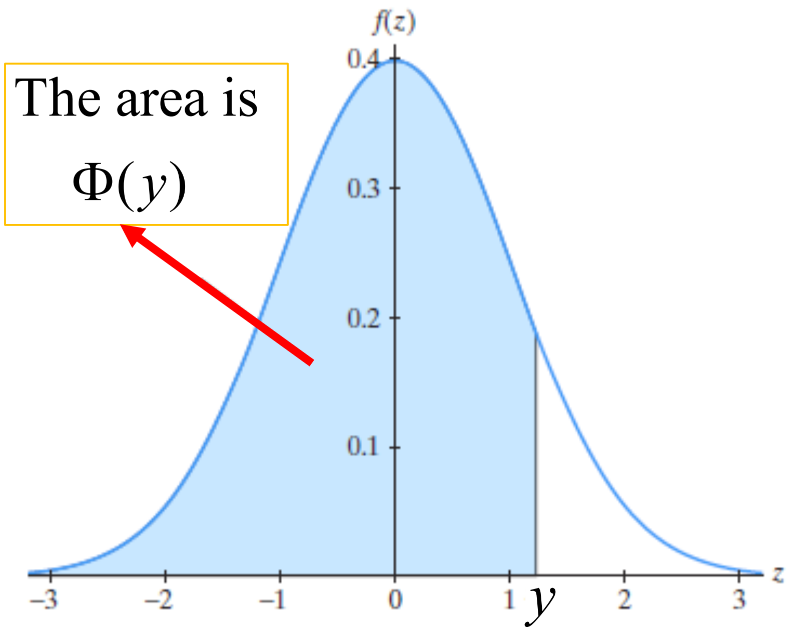
Hence we get its mgf $M(t) = \exp(-7t + 8t^2).$

Definition [Standard normal distribution]

Y is said to be a **standard normal distribution** if $Y \sim N(0,1)$.

$$\Leftrightarrow f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

Its cdf $\Phi(y) = P(Y \le y) = \int_{-\infty}^{y} f(z)dz = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$



- Values of $\Phi(y)$ for some values of $y \ge 0$ are given in Appendix B in our textbook! (Page 502)

You can think why there is only positive numbers in that table.

- Due to the ***symmetry*** of $f(y)$,

$$\Phi(y) = 1 - \Phi(y)$$

for all real y.

Example 2 (Page 116)

$Z \sim N(0,1)$ Then compute :

$P(Z \le 1.24)$、$P(1.24 \le Z \le 2.37)$、$P(-2.37 \le Z \le -1.24)$、

$P(Z > 1.24)$、$P(Z \le -2.14)$、$P(-2.14 \le Z \le 0.77)$.

Solution :

Using Table V_a in Appendix B, we have:

$$P(Z \le 1.24) = \Phi(1.24) = 0.8925$$

$$P(1.24 \le Z \le 2.37) = \Phi(2.37) - \Phi(1.24) = 0.9911 - 0.8925 = 0.0986$$

$$P(-2.37 \le Z \le -1.24) = P(1.24 \le Z \le 2.37) = 0.0986.$$

Using Table V_b in Appendix B, we have:

$$P(Z > 1.24) = 0.1075$$

$$P(Z \le -2.14) = P(Z \ge 2.14) = 0.0162$$

Using both table, we have:

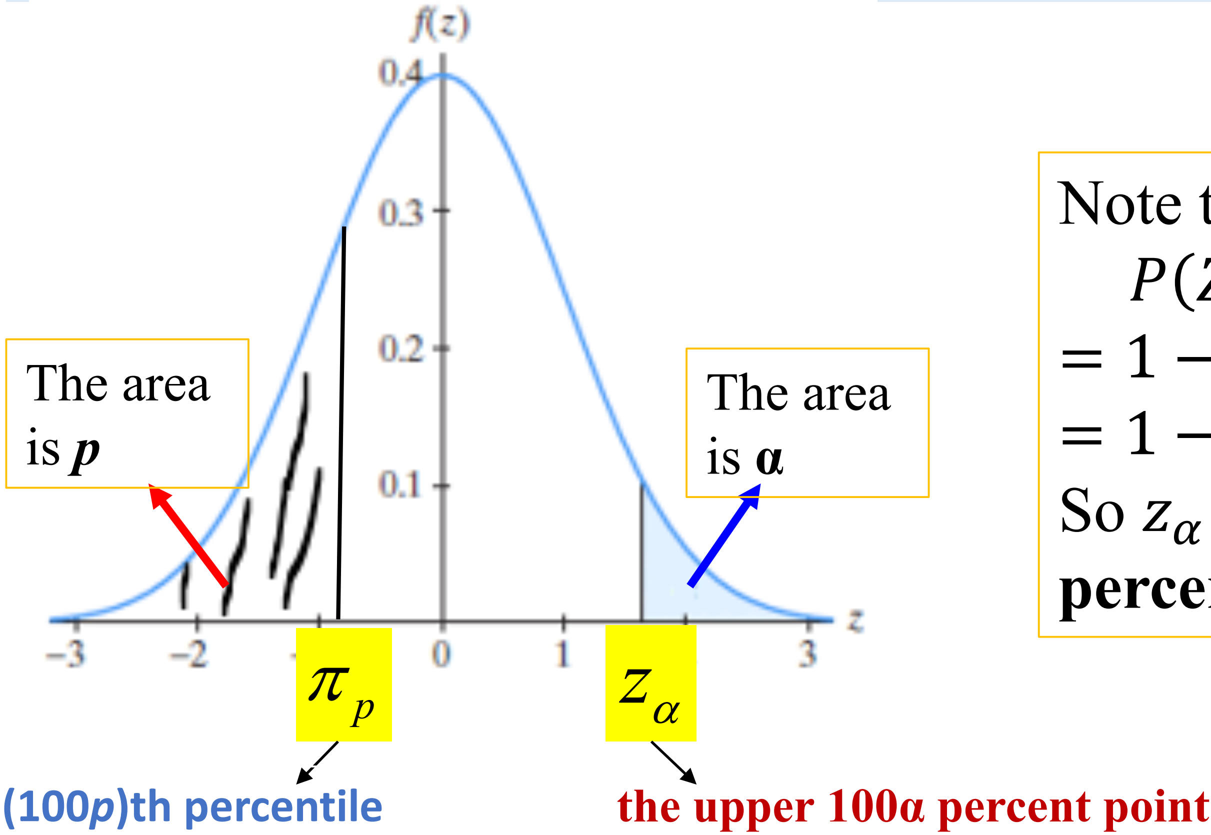
$$P(-2.14 \le Z \le 0.77) = P(Z \le 0.77) - P(Z \le -2.14) = 0.7794 - 0.0162 = 0.7632.$$

given a probability *p*, we can also find a constant *a* so that $P(Z \le a) = p$ through using the table!.

Definition [the upper 100α percent point]

It is a number z_α such that the area **under $f(x)$ to the right of z_α** is $\boldsymbol{\alpha}$. That is,

$$P(Z \geq z_\alpha) = \alpha$$



Note that

$$P(Z < z_\alpha)$$
$$= 1 - P(Z \geq z_\alpha)$$
$$= 1 - \alpha.$$

So z_α is the **(100*(1-α)*)th percentile**.

$P(X \leq \pi_p) = p, \pi_p$ is (100p)th percentile.

Example 3 (Page 117)

$Z \sim N(0,1)$, Find $z_{0.0125}$、$z_{0.05}$、$z_{0.025}$.

Solution :

$\Leftrightarrow P(Z \geq z_{0.0125}) = 0.0125.$ By checking the table V_b, $z_{0.0125} = 2.24.$

Similarly, $z_{0.05} = 1.645$, $z_{0.025} = 1.960.$

Now we know to compute $\Phi(y)$ by looking up the table for $Y \sim N(0,1)$. But what if Y is not standard normal?

Theorem 3.3-1

If Y is $N(\mu, \sigma^2)$, then $X = (Y - \mu)/\sigma$ is $N(0,1)$.

Proof : The idea is to show X has the same cdf as $N(0,1)$.

$$P(X \leq x) = P\left(\frac{Y-\mu}{\sigma} \leq x\right) = P(Y \leq \sigma x + \mu) = \int_{-\infty}^{\sigma x + \mu} f(y)dy$$

$$= \int_{-\infty}^{\sigma x + \mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(y-\mu)^2}{\sigma}\right)dy$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}w^2\right)dw = \Phi(x)$$

Change of variable with $w = \dfrac{y-\mu}{\sigma}$

cdf of N(0,1)

With the theorem just now, for $X \sim N(\mu,\sigma^2)$,

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right),$$

where $\Phi(\bullet)$ is the cdf of $N(0,1)$.

Example 4 (Page 118)

$X \sim N(3,16)$. Compute $P(4 \le X \le 8)$ and $P(0 \le X \le 5)$.

Solution :

$$P(4 \le X \le 8) = P(\frac{4-3}{4} \le \frac{X-3}{4} \le \frac{8-3}{4}) = \Phi(1.25) - \Phi(0.25) = 0.8944 - 0.5987 = 0.2957.$$

$$P(0 \le X \le 5) = P(\frac{0-3}{4} \le \frac{X-3}{4} \le \frac{5-3}{4}) = \Phi(0.5) - \Phi(-0.75) = 0.6915 - 0.2266 = 0.4649.$$

In the next theorem, we give a relationship between the chi-square and normal distributions.

Theorem 3.3-2

If the $RV\ X$ is $N(\mu,\sigma^2)$ with $\sigma^2>0$, then $\dfrac{(X-\mu)^2}{\sigma^2}\sim\chi^2(1)$.

Proof: Let $V=Z^2=\dfrac{(X-\mu)^2}{\sigma^2}$. Then consider the *cdf* of V:

$$G(v)=P(V\le v)=P(-\sqrt{v}\le Z\le\sqrt{v}) \text{ with } Z=\frac{X-\mu}{\sigma}, v\ge 0.$$

$$G(v)=\int_{-\sqrt{v}}^{\sqrt{v}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz=2\int_0^{\sqrt{v}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

$$G(v)=2\int_0^{v}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y}\frac{1}{2\sqrt{y}}dy=\int_0^{v}\frac{1}{\sqrt{2\pi y}}e^{-\frac{1}{2}y}dy,\qquad v\ge 0.$$

Changing of variable with $z=\sqrt{y}$ and $\dfrac{dz}{dy}=\dfrac{1}{2\sqrt{y}}$:

The *pdf* of V is: $g(v)=G'(v)=\dfrac{1}{\sqrt{2\pi v}}e^{-\frac{1}{2}v}, v\ge 0.$ since $g(v)$ is a *pdf*, $\int_0^\infty g(v)dv=1$.

$$\Rightarrow 1=\int_0^\infty\frac{1}{\sqrt{2\pi v}}e^{-\frac{1}{2}v}dv=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{1}{\sqrt{x}}e^{-x}dx=\frac{1}{\sqrt{\pi}}\int_0^\infty x^{1/2-1}e^{-x}dx=\frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2})$$

$$\Rightarrow \Gamma(\frac{1}{2})=\sqrt{\pi}\Rightarrow g(v)=\frac{1}{\Gamma(\frac{1}{2})2^{1/2}}v^{1/2-1}e^{-\frac{1}{2}v},\qquad v>0.$$

$$\Rightarrow V\sim\chi^2(1)$$