Discrete Distribution（离散分布）

Starting from this section, some typical random experiments and corresponding distribution will be introduced.

Section 2.4 *Binomial distribution*

- Bernoulli experiment

  The outcome can be classified in one of two ***mutually exclusive and exhaustive*** ways--say either *success* or *failure*. (e.g. *female* or *male*; *life* or *death*)

- Bernoulli trials

  When a Bernoulli experiment is performed several *independent* times and the probability of *success*—say, p—remains the *same* from trial to trial. In other words, we let p donate the probability of *success* on each trial. And we define $q \triangleq 1 - p$ to donate the probability of *failure*.

trial—试验

Example 1:

successive—连续的

You are a fan of lottery. For a lottery, the probability of winning is $\frac{1}{1000}$. If you buy the lottery for 10 successive days, that corresponds to 10 *Bernoulli trials* with $p=\frac{1}{1000}$.

Assuming independence

- Bernoulli distribution

Let X be a RV associated with a Bernoulli trial with the probability of success p.

RV: X: S→X(S)⊆R, S={success, failure}. And define:

X(success)=1, X(failure)=0, X(S)={0, 1}

The pmf of X could be written as:

$f(x)$: X(S) = {0,1} →[0,1], $f(x)=p^x(1-p)^{1-x}$.

Then we say X has a **Bernoulli distribution** whose characteristic:

$$E(X)=\sum_{x\in X(S)} xf(x)=0\cdot(1-p)+1\cdot p=p$$

$$Var(X)=E\left[\left(X-E(X)\right)^2\right]=\sum_{x\in X(S)}(x-p)^2 f(x)=p^2(1-p)+(1-p)^2 p=(1-p)p=pq$$

$$M(t)=E(e^{tX})=\sum_{x\in X(S)} e^{tx} f(x)=e^{t\cdot 0}\times(1-p)+e^{t\cdot 1}\times p=(1-p)+pe^t$$

- In a sequence of n Bernoulli trials, we shall let X_i denote the Bernoulli random variable associated with the ith trial. An observed sequence of n Bernoulli trials will then be an n-tuple of zeros and ones, and we often call this *collection* a **random sample** of size n from a Bernoulli distribution

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Example 2 (Page 74)

Out of millions of instant lottery tickets, suppose that 20% are winners. If 5 tickets are purchased, then (0, 0, 0, 1, 0) is a ***random sample***. Assuming *independence* between purchasing different tickets, the probability of this sample is

$$p = (0.2)(0.8)^4.$$

Multiplication Rule:

If $A_1, A_2, \ldots, A_n$ are *mutually independent*,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

- Binomial distribution

We are interested in the *number of successes* in n Bernoulli trials, the order of the occurrence is not concerned.

A binomial experiment satisfies the following properties:

1. A Bernoulli (success–failure) experiment is performed n times.
2. The trials are independent.
3. The probability of success on each trial is a constant p; the probability of failure is $q=1-p$.
4. The random variable X equals the number of successes in the n trials.

$P(A \cap B) = P(A)P(B)$

$\rightarrow$ multiplicate rule of probability.

$X : S \to X(S);$ $\qquad X(S) = \{0, 1, 2, \ldots, n\}$

If $x \in X(S)$, successes occurs, the number of ways of selecting x successes in n trials is $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

Since trials are independent, the probability of each way is

$$p^x (1-p)^{n-x}$$

Why *pmf* well-defined?

$f(x) > 0$

$\sum_{x \in X(S)} f(x) = 1$

$P(x \in A) = \sum_{x \in A} f(x)$

$\Rightarrow$ *pmf* of X : $f(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \ldots, n.$

$f(x) \to$ Binomial probability, X is also said to have a Binomial distribution and denoted by $X \sim b(n, p)$

Example 2 (revisited)

If X is the number of winning tickets among n=5 that are produced, then the probability of purchasing 2 winning tickets is

$$f(2) = P(X = 2) = \binom{5}{2} (0.2)^2 (0.8)^3, \qquad X \sim b(5, 0.2)$$

- cdf of Binomial distribution

Assume X have a Binomial distribution b(n,p), the cdf of X is

$$F(x) = P(X \le x) = \sum_{y \in \{n | n \le x\}} f(y) = \sum_{y=0}^{\lfloor x \rfloor} \binom{n}{y} p^y (1-p)^{n-y}$$

$x \in (-\infty, +\infty)$, where $\lfloor x \rfloor$ is the greatest integer that $\le x$

Example 3

Chickens are raised for laying eggs. Let p = 0.5 be the probability that the newly hatched chick is a female. Assuming independence, let X be the number of female chicks out of 10 newly hatched chicks selected at random.

$Obviously, X \sim b(10, 0.5);$ $compute$ $P(X \le 5)$、$P(X = 6)$、$P(X \ge 6)$

Solution:

$$P(X \le 5) = \sum_{x=0}^{5} \binom{10}{x} (0.5)^x (0.5)^{n-x}; \quad P(X \ge 6) = 1 - P(X \le 5).$$

$$P(X = 6) = f(6) = \binom{10}{x} (0.5)^6 (0.5)^4$$

➢ mgf of Binomial distribution

Assume X have a Binomial distribution b(n,p), the mgf of X is

$$M(t)=E(e^{tX})=\sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x}$$

$$=\sum_{x=0}^{n}\binom{n}{x}(pe^t)^x(1-p)^{n-x}$$

$$=\left[(1-p)+pe^t\right]^n, \qquad -\infty<t<+\infty.$$

From the expansion of

$$(a+b)^n=\sum_{x=0}^{n}\binom{n}{x}a^x b^{n-x}$$

with $\begin{cases} a=pe^t \\ b=1-p \end{cases}$

- $M'(t)=n\left[(1-p)+pe^t\right]^{n-1}pe^t \quad \Rightarrow M'(0)=E(X)=np.$
- $M''(t)=n(n-1)\left[(1-p)+pe^t\right]^{n-2}p^2e^{2t}+n\left[(1-p)+pe^t\right]^{n-1}pe^t$

$\Rightarrow M''(0)=E(X^2)=n(n-1)p^2+np.$

- $Var(X)=E(X^2)-\left[E(X)\right]^2=\left[n(n-1)p^2+np\right]-n^2p^2=np(1-p)$

By the way, when n=1, the ***Binomial distribution*** reduces to ***Bernoulli distribution***.

Section 2.5 *Negative Binomial distribution*

We are interested in the situation that we observe a sequence of independent Bernoulli trials until exactly r successes occur, where r is a fixed positive integer.
We define a RV X *to* denote the trial number on which the rth success is observed:

$X: S \to X(S) = \{r, r+1, \ldots\ldots\}$, let $f(x)$ denote the *pmf* of X.

Then
$$f(x) = P(\{\text{At the } x\text{th trial, } r\text{th success is observed}\})$$
$$= P(\{\text{for the first } x-1 \text{ trials, } r-1 \text{ successes have been observed}\}$$
$$\cap \{\text{At the } x\text{th trial, the outcome is success}\})$$
$$= P(A \cap B) = P(A)P(B) \quad (\text{because } A \text{ and } B \text{ are independent})$$

def A

def B

$$P(A) = \binom{x-1}{r-1} p^{r-1}(1-p)^{x-r}, \quad P(B) = p$$

$$\Rightarrow f(x) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r}, \quad x = r, r+1, \ldots\ldots$$

In the case $r = 1, f(x) = p(1-p)^{x-1},\ x = 1, 2, \ldots$

For integer k, $P(X > k) = \sum_{x=k+1}^{\infty} p(1-p)^{x-1} = \dfrac{p(1-p)^k}{1-(1-p)} = (1-p)^k.$

$$P(X \le k) = \sum_{x=1}^{k} p(1-p)^{x-1} = 1 - P(X > k) = 1 - (1-p)^k$$

Example 1 (Page 83)

Biology students are checking eye color of fruit flies.
For individual fly, P(white)=1/4, P(red)=3/4.
Assume the observations are independent Bernoulli trials:

At least 4 flies. $P(X \ge 4) = P(X > 3) = (1 - 1/4)^3 = (3/4)^3$

At most 4 flies. $P(X \le 4) = 1 - (1 - 1/4)^4$

4 flies. $P(X = 4) = (1/4)(3/4)^3$

➢ Mean and Variance

- direct calculation
- $mgf \quad M(t) = E(e^{tX})$

$$prove: E(X) = \sum_{x \in X(S)} xf(x) = \sum_{x=r}^{\infty} xf(x) = \frac{r}{p}$$

$$prove: Var(X) = E(X^2) - \left[E(X)\right]^2 = \frac{r(1-p)}{p^2}$$

proof : The *mgf* of X is

$$\text{M(t)=E(e}^{tX}\text{)=}\sum_{x=r}^{\infty} e^{tx} \binom{x-1}{r-1} p^r (1-p)^{x-r} = (pe^t)^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} \left[(1-p)e^t\right]^{x-r}$$

$$= \frac{(pe^t)^r}{\left[1-(1-p)e^t\right]^r}, \qquad where\ (1-p)e^t < 1.$$

$$\Rightarrow M'(t) = r(pe^t)^r \left[1-(1-p)e^t\right]^{-r-1};$$

$$M''(t) = r(pe^t)^r(-r-1)\left[1-(1-p)e^t\right]^{-r-2}\left[-(1-p)e^t\right]$$

$$+ r^2(pe^t)^{r-1}(pe^t)\left[1-(1-p)e^t\right]^{-r-1}$$

$$\Rightarrow E(X) = M'(0) = rp^{-1}; \quad E(X^2) = M''(0) = rp^{-2}(r+1-p).$$

$$\Rightarrow Var(X) = E(X^2) - \left[E(X)\right]^2 = rp^{-2}(r+1-p) - r^2p^{-2} = rp^{-2}(1-p).$$

Section 2.6 *Poisson Distribution*

There are experiments that result in counting the number of times particular events occur at given times or with given physical objects.

E.g.

- The number of flaws in a 100 feet long wire
- The number of customers that arrive at a ticket window between 9p.m. to 10p.m.

Counting such events can be looked upon as observations of a ***random variable*** associated with an ***approximate Poisson process(APP)***, provided that the conditions in the following definition are satisfied.

Definition 2.6-1 [App]

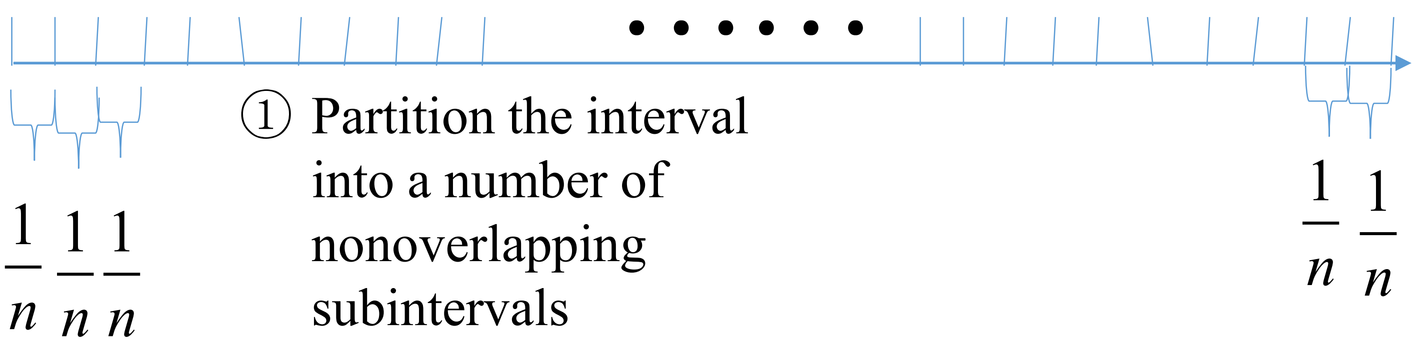
Let the number of occurrences of some event in a given continuous interval be counted. Then we have an **APP** with parameter $\lambda > 0$ if

(a) The numbers of occurrences in nonoverlapping subintervals are **independent**.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately $\boldsymbol{\lambda h}$.

(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially **0**.

Consider a random experiment desired by App. Let X denote the number of occurrences in an interval of length 1. We aim to find an approximation for $P(X=x)$ where x is a nonnegative integer.



② If n is sufficiently large($n \gg x$), $P(X=x)$ can be approximated by the probability that exactly x of these n subintervals each has one occurrence.

③

I. By condition (b), the probability of one occurrence in anyone subinterval of length $1/n$ is approximately λ/n.

II. By condition (c), the probability of 2 or more occurrences in any one subinterval is essentially 0. That is, For each subinterval there is either no occurrence or one occurrence. [The probability of occurrence is $\frac{\lambda}{n}$.

I. ℘ II. ⇒ The occurrence and nonoccurrence in each interval could be treated as Bernoulli trials.

The number of occurrence ~ $b(1, \frac{\lambda}{n})$.

III. By condition (a), we have a sequence of n Bernoulli trials with probability p approximately equal to $\frac{\lambda}{n}$.

④ Therefore, $P(X=x)$ can be approximated by the binomial probability:

$$P(X=x) \approx \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

⑤ If let n→ ∞, then

$$\lim_{n\to\infty}\frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty}\frac{n!}{(n-x)!n^x}\cdot\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

Now for fixed n, we have:

$$\lim_{n\to\infty}\frac{n!}{(n-x)!n^x} = \lim_{n\to\infty}\frac{n(n-1)\cdots(n-x+1)}{n^x} = \lim_{n\to\infty}\left[1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right] = 1,$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}, \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$$

We have $P(X=x) = \lim_{n\to\infty}\frac{n!}{(n-x)!n^x}\cdot\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} = \frac{\lambda^x e^{-\lambda}}{x!}$.

Since we know $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$,

we replace x with $-\lambda$.

Definition 2.6-2 [Poisson distribution]

It can be verified that

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

is a well-defined pmf. If a RV X has $f(x)$ as its *pmf*, then X is said to have a **Passion distribution**.

What is the interpretation of λ ?

- Mean and Variance

The *mgf* of a Poisson deitribution RV X is

$$M(t)=E(e^{tX})=\sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

Maclaurin expansion

$$e^{\lambda} = \sum_{x=0}^{\infty}\frac{\lambda^x}{x!}$$

$$M'(t) = \lambda e^t e^{\lambda(e^t-1)} \Rightarrow M'(0) = \lambda = E(X)$$

$$M''(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)} \Rightarrow M''(0) = \lambda + \lambda^2 = E(X^2)$$

$$\Rightarrow E(X) = \lambda$$

$$Var(X) = E(X^2) - \left[E(X)\right]^2 = \lambda + \lambda^2 - \lambda^2 = \lambda.$$

λ is the average number or variance of occurrences in the interval.

Example 1 (Page 91)

In a large city, telephone calls to 110 come on the average of 2 every 3 minutes. If one models with App, what is the probability of five or more calls arriving in a 9-minute period?

Solution: Let X denote the number of calls in a 9-*minute* period.

$$\text{Then } E(X) = 6 = \lambda \Rightarrow f(x) = \frac{6^x e^{-6}}{x!}$$

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^{4} \frac{6^x e^{-6}}{x!} = 0.$$