Chapter 5 Distributions of Functions of random variables

Section 5.1 Functions of one random variable

Question : Let X be a RV of either discrete or continuous type. Consider a function of X, say, $Y = u(X)$. Then Y is also be a RV and has its *pmf* or *pdf*.

How to compute the *pmf* or *pdf* ?

- Discrete case:

Let X be a discrete RV with pmf $f(x): \overline{S} \to [0,1]$, and $Y = u(X)$ be a one-to-one transformation with inverse $X = v(Y)$. Then the *pmf* of Y is

$g(Y) = P(Y = y) = P(u(x) = y) = P(x = v(y))$ for $y \in u(\overline{S})$

since $P(X = x) = f(x)$, we have $g(y) = f\left[v(y)\right]$ for $y \in u(\overline{S})$.

Example1 (Page 177)

Let X has a Possion distribution with $\lambda = 4$, so it has the *pmf*

$$f(x) = \frac{4^{-x} e^{-4}}{x!}, \qquad x = 0,1,2,\ldots$$

If Y=$\sqrt{X}$, what's the *pmf* $g(y)$ of Y?

Example1 (c.n.t.)

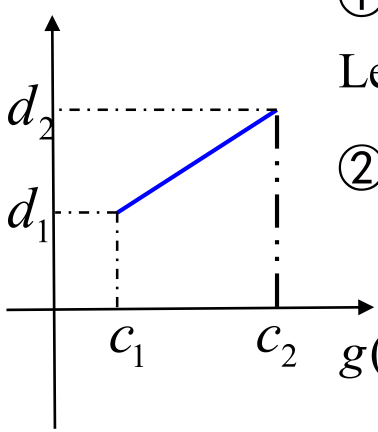
$Solution: Y = u(X) = \sqrt{X} \Rightarrow X = v(Y) = Y^2$

$$g(y) = P(Y = y) = P(X = y^2) = f(y^2) = \frac{4^{y^2} e^{-4}}{(y^2)!}, \qquad y = 0, \sqrt{1}, \sqrt{2}, \dots$$

➢ Continuous case:

Let X be a continuous-type RV with pdf $f(x):[c_1, c_2] \to [0, +\infty)$

• **Case 1 :** $Y = u(X)$ is a continuous increasing function of X with inverse function $X = v(Y)$. To calculate the pdf of Y, say, $g(y)$,



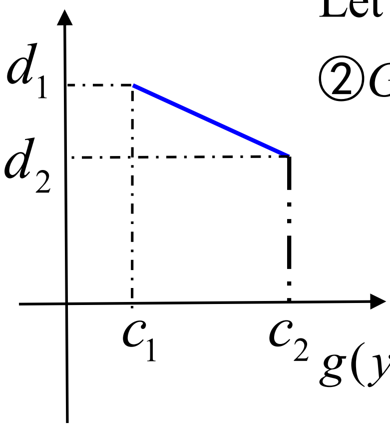
①Determine the range of Y : Since $Y = u(X)$ is continuousand increasing, Let $d_1 = u(c_1)$ and $d_2 = u(c_2)$. Then $d_1 \le Y \le d_2$

②$G(y) = P(Y \le y) = P[u(X) \le y] = P[X \le v(y)] = \int_{c_1}^{v(y)} f(x)dx$

$$g(y) = G'(y) = \frac{dG(y)}{d[v(y)]} \frac{d[v(y)]}{dy} = \frac{d\left[\int_{c_1}^{v(y)} f(x)dx\right]}{d[v(y)]} \frac{d[v(y)]}{dy} = f[v(y)]v'(y)$$

$$= f[v(y)] \frac{d[v(y)]}{dy} = f[v(y)] \left| \frac{d[v(y)]}{dy} \right|. \qquad d_1 \le y \le d_2$$

• **Case 2 :** $Y = u(X)$ is a continuous decreasing function of X with inverse function $X = v(Y)$. To calculate the pdf of Y, say, $g(y)$,



①Determine the range of Y : Since $Y = u(X)$ is continuousand increasing,

Let $d_1 = u(c_1)$ and $d_2 = u(c_2)$. Then $d_2 \le Y \le d_1$

②$G(y) = P(Y \le y) = P\left[u(X) \le y\right] = P\left[X \ge v(y)\right] = 1 - P\left[X < v(y)\right]$

$$= 1 - \int_{c_1}^{v(y)} f(x)dx$$

$$g(y) = G'(y) = \frac{dG(y)}{d\left[v(y)\right]}\frac{d\left[v(y)\right]}{dy} = \frac{d\left[1 - \int_{c_1}^{v(y)} f(x)dx\right]}{d\left[v(y)\right]}\frac{d\left[v(y)\right]}{dy} = -f\left[v(y)\right]v'(y)$$

$$= -f\left[v(y)\right]\frac{d\left[v(y)\right]}{dy} = f\left[v(y)\right]\left|\frac{d\left[v(y)\right]}{dy}\right|. \qquad d_2 \le y \le d_1$$

Summary : In both increasing or decreasing cases, $g(y) = f\left[v(y)\right]\left|\frac{d\left[v(y)\right]}{dy}\right|$.

Example2 (Page 174)

Let X has the pdf $f(x) = 3(1 - x^2),\ 0 < x < 1$. Consider $Y = (1 - X)^3$

Calculate the *pdf* of Y, $g(y)$.

$Y = u(X) = (1-X)^3 \rightarrow$ continuous decreasing function.

Inverse function: $X = v(Y) = 1 - Y^{1/3}$.

①Determine the range of Y: Since $0 < x < 1$, we have $0 < y < 1$.

②$g(y) = f\left[v(y)\right]\left|\dfrac{d\left[v(y)\right]}{dy}\right|$ where $\dfrac{d\left[v(y)\right]}{dy} = -\dfrac{1}{3}y^{-2/3}$

$$= 3\left[1-(1-y^{1/3})\right]^2\left|-\frac{1}{3}y^{-2/3}\right| = 1, \qquad 0 < y < 1. \qquad Y \sim U(0,1).$$

Theorem 5.1-1 [Page175. random number generator]

Let $Y \sim U(0, 1)$. $F(x)$ have the properties of a cdf of continuous type with $F(a) = 0$, $F(b) = 1$, and suppose that $F(x)$ is strictly increasing such that $F(x)$:$(a, b)\rightarrow[0,1]$ where a and b could be $-\infty$ and ∞ respectively. Then $X = F^{-1}(Y)$ is a continuous-type RV with cdf $F(x)$.

Proof. Idea: We need to show $P(X \le x) = F(x)$.

$$P(X \le x) = P(F^{-1}(Y) \le x) = P(Y \le F(x)) \qquad as \ \left\{y \middle| F^{-1}(y) \le x\right\} = \left\{y \middle| y \le F(x)\right\}.$$

Note that $Y \sim U(0,1)$, $P(Y \le y) \overset{0<y<1}{=} \int_0^y 1dz = y$. *Therefore,*

$$P(X \le x) = P(Y \le F(x)) = F(x)$$ ← Complete the proof.

Remark : Random number generator from arbitary distribution ——

① gnerator a random number from $U(0,1)$ ②Take $x = F^{-1}(y)$.

Theorem 5.1-2

Let X have the cdf $F(x)$ of the continuous type that is strictly increasing on the support $a < x < b$. Then the random variable Y, defined by $Y = F(X)$, has a distribution that is $U(0, 1)$.

Proof. Since $F(a) = 0$ and $F(b) = 1$, and $F(x)$ is strictly increasing,

$Y = F(X)$ with range $\overline{S_Y} = (0,1)$.

Consider the cdf of $Y: P(Y \le y) = P(F(x) \le y),\qquad y \in (0,1)$

Since $F(x)$ is strictly increasing,

$\{F(X) \le y\}$ is equivalent to $\{X \le F^{-1}(y)\}$; thus,

$P(Y \le y) = P[F(X) \le y] = P(X \le F^{-1}(y))$

Since $P(X \le x) = F(x)$, we have

$P(Y \le y) = F[F^{-1}(y)] = y,\ 0 < y < 1,$

which is the cdf of a $U(0, 1)$ random variable.

- What if the Y=u(X) is not one-to-one

Example3

Assume X is a continuous RV with pdf $f(x)=\dfrac{1}{\pi(1+x^2)}$. $\quad x\in(-\infty,+\infty)$.

Let $Y=X^2$. $\quad$ Find the pdf of Y. $\overline{S_Y}=[0,+\infty)$.

Let the cdf of Y be $G(y)$. Then $G(y)=P(Y\le y)$, $\quad y\in[0,+\infty)$.

$$=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})=\int_{-\sqrt{y}}^{\sqrt{y}}f(x)dx$$

$$=\int_{-\sqrt{y}}^{\sqrt{y}}\frac{1}{\pi(1+x^2)}dx=2\int_0^{\sqrt{y}}\frac{1}{\pi(1+x^2)}dx$$

$$\Rightarrow g(y)=G'(y)=\frac{2}{\pi(1+y)}\frac{1}{2}\frac{1}{\sqrt{y}}=\frac{1}{\pi(1+y)\sqrt{y}}.$$

①Find the cdf $G(y)=P(Y\le y)$

②Get the pdf $g(y)=G'(y)$

Chapter 5 Distributions of Functions of random variables

Section 5.2 Several Random Variables (Multivariate RVs)

- Random experiment : Any procedure that can be repeated infinitely times and has more than one possible outcomes

- Performing a random experiment one time, the outcome may contain:

* One thing of interest univariate *RV: X*, $f(x)$, joint pmf or pdf
* A tuple of two things of interest bivariate *RV:* (X,Y), $f(x,y)$, pmf or pdf
* A tuple of several things of interest Multivariate *RV*: $(X_1, X_2, \ldots\ldots, X_n)$

The corresponding joint pmf $f(x_1, x_2, \ldots, x_n)$ with domain $\overline{S}$

Discrete type

① $f(x_1, x_2, \ldots, x_n) \geq 0$

② $\displaystyle\sum_{(x_1, x_2, \ldots, x_n) \in \overline{S}} f(x_1, x_2, \ldots, x_n) = 1$

③ $\displaystyle P((x_1, x_2, \ldots, x_n) \in A) = \sum_{(x_1, x_2, \ldots, x_n) \in A} f(x_1, x_2, \ldots, x_n)$

Continuous type

① $f(x_1, x_2, \ldots, x_n) \geq 0$

② $\displaystyle\int \cdots \int_{\overline{S}} f(x_1, x_2, \ldots, x_n) dx_1 \cdots dx_n = 1$

③ $\displaystyle P((x_1, x_2, \ldots, x_n) \in A) = \int \cdots \int_A f(x_1, x_2, \ldots, x_n) dx_1 \cdots dx_n$

A random experiment consists of performing a random experiment several times independently. For this case, the joint pmf or pdf is easier to be obtained.

Example 1

Roll a fair die twice let X_1 denote the point of the first roll and X_2 denote the point of the second roll.

$$\text{For } X_1 = x_1, \text{ its pmf } f_1(x_1) = P(X_1 = x_1) = \frac{1}{6}, \quad x_1 = 1, 2, \ldots, 6$$

$$\text{For } X_2 = x_2, \text{ its pmf } f_2(x_2) = P(X_2 = x_2) = \frac{1}{6}, \quad x_2 = 1, 2, \ldots, 6$$

Assuming the two experiments (X_1 and X_2) ar independent. Then for $X_1 = x_1, X_2 = x_2$, the joint pmf of X_1 and X_2 is $f(x_1, x_2) = P(X_1 = x_1, X_2 = x_2) = f_1(x_1) f_2(x_2)$.

Definition 5-2-1

[Random Sample of size 2 from a common distribution]

Repeat an experiment twice and independently leads to two independent RVs from the same distribution

Definition 5-2-2[n independent RVs]

The n RVs $X_1, X_2, \ldots, X_n$ are said to be mutually independent if $f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$, where $f(x_1, x_2, \ldots, x_n)$ is the joint pmf or pdf of $x_1, x_2, \ldots, x_n$, and $f_i(x_i)$ for $n = 1, 2, \ldots, n$ is the marginal pmf or pdf of X_i.

In this case suppose the pmf or pdf is $f(x_i),\ i = 1, \ldots, n$

$\Rightarrow$ The joint pdf or pmf is given by

$$g(x_1, \ldots, x_n) = f(x_1) \cdots f(x_n)$$

Definition 5-2-3

[Random Sample of size n from a common distribution]

Repeat an experiment n times and independently leads to n independent and identically distributed RVs $X_1, X_2, \ldots \ldots, X_n$

Example 2 [Page 190]

"independent and identically distributed" is often be written as i.i.d.

Let X_1, X_2, X_3 be a random samle of size 3 from a distribution with pdf $f(x) = e^{-x}, \quad x \in (0, +\infty)$

The joint pdf $g(x_1, x_2, x_3) = f(x_1) f(x_2) f(x_3) = e^{-x_1} e^{-x_2} e^{-x_3}$.

Question: $\quad P(0 < x_1 < 1,\ 2 < x_2 < 4,\ 3 < x_3 < 7) \quad x_i \in (0, +\infty),\ i = 1, 2, 3$

Solution: $\quad P(0 < x_1 < 1,\ 2 < x_2 < 4,\ 3 < x_3 < 7)$

$$= P(0 < x_1 < 1) P(2 < x_2 < 4) P(3 < x_3 < 7) = \int_0^1 e^{-x_1} dx_1 \int_2^4 e^{-x_2} dx_2 \int_3^7 e^{-x_3} dx_3$$

Definition 5-2-4 [Mathematical expectation for n RVs]

For n RVs, $X_1, X_2, \ldots, X_n$ where the joint pmf or pdf is represented by

$f(x_1, x_2, \ldots, x_n),\ (x_1, x_2, \ldots, x_n) \in \overline{S}.$ For a function $u(x_1, x_2, \ldots, x_n)$,

its mathematical expectation is given by

$$E\left[u(x_1, x_2, \ldots, x_n)\right] = \begin{cases} \displaystyle\sum_{(x_1, x_2, \ldots, x_n) \in \overline{S}} u(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n) & \text{Discrete RVs} \\ \displaystyle\int \cdots \int_{\overline{S}} u(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n) dx_1 \cdots dx_n & \text{Continuous RVs} \end{cases}$$

In case where the n RVs $X_1, X_2, \ldots, X_n$ are independent, $f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

Mathematical Expectation is a linear operator

The next theorem proves that the expected value of the product of functions of n independent random variables is the product of their expected values.

Theorem 5.2-1

Assume $X_1, X_2, \ldots, X_n$ are independent RVs and $Y = u_1(X_1) u_2(X_2) \cdots u_n(X_n)$

If $E\left[u_i(X_i)\right]$ $i = 1, 2, \ldots, n$ exist. Then

$$E(Y) = E\left[u_1(X_1) u_2(X_2) \cdots u_n(X_n)\right] = E\left[u_1(X_1)\right]\left[u_2(X_2)\right] \cdots \left[u_n(X_n)\right]$$

Proof of theorem 5.2-1

Proof. In the discrete case(the continuous case is left as an exercise)

$$E[u_1(X_1)u_2(X_2)\cdots u_n(X_n)] = \sum_{(x_1,x_2,\ldots,x_n)\in\bar{S}} u_1(x_1)u_2(x_2)\cdots u_n(x_n) f(x_1,x_2,\ldots,x_n)$$

$$= \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} u_1(x_1)u_2(x_2)\cdots u_n(x_n) f_{X_1}(x_1)\cdots f_{X_n}(x_n)$$

$$= \sum_{x_1} u(x_1)f(x_1)\sum_{x_2} u(x_2)f(x_2)\cdots\sum_{x_n} u(x_n)f(x_n)$$

$$= E[u_1(X_1)][u_2(X_2)]\cdots[u_n(X_n)]$$

We now prove an important theorem about the mean and the variance of a ***linear combination*** of random variables.

Theorem 5.2-2

Assume $X_1, X_2, \ldots, X_n$ are independent RVs with respective mean $u_1, u_2, \cdots u_n$

and variances $\sigma_1^2, \ldots, \sigma_n^2$.Consider $Y = \sum_{i=1}^{n} a_i X_i$ where $a_1, \ldots, a_n$ are real constants.

Then $E(Y) = \sum_{i=1}^{n} a_i u_i,\; Var(Y) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

Proof of theorem 5.2-2

Proof.

$$E(Y)=E(\sum_{i=1}^{n}a_iX_i)=\sum_{i=1}^{n}a_iE(X_i)=\sum_{i=1}^{n}a_iu_i$$

By the theorem we discussed before

$$Var(Y)=E\left\{[Y-E(Y)]^2\right\}=E\left\{\left[\sum_{i=1}^{n}a_iX_i-\sum_{i=1}^{n}a_iu_i\right]^2\right\}=E\left\{\left[\sum_{i=1}^{n}a_i(X_i-u_i)\right]^2\right\}$$

$$=E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}a_ia_j(X_i-u_i)(X_j-u_j)\right]$$

$$=\sum_{i=1}^{n}a_i^2\sigma_i^2.$$

When $i=j$:

$$E\left[(X_i-u_i)(X_j-u_j)\right]=\sigma_i^2$$

When $i\neq j$:

$$E\left[(X_i-u_i)(X_j-u_j)\right]=0$$

Example 3

When $X_1, X_2, \ldots, X_n$ are independent and identically distributed RV with mean u and variance σ^2. Consider $\overline{X}=\sum_{i=1}^{n}\frac{1}{n}X_i$.

Sample mean

$$E(\overline{X})=\sum_{i=1}^{n}\frac{1}{n}u=u. \qquad Var(\overline{X})=\sum_{i=1}^{n}\left(\frac{1}{n}\right)^2\sigma^2=\frac{\sigma^2}{n}.$$

Definition 5.2-5 [Statistic]

Any function of the sample $X_1, X_2, \ldots, X_n$ that don't have any unknown parameters is called a statistic.

Here $\overline{X}$ is a statistic and also an estimator of mean μ.

Another important statistic is the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

and later we find that S^2 is an estimator of σ^2.

Chapter 5 Distributions of Functions of random variables

Section 5.4 Moment generating function technique

➢ Mgf, if exists, uniquely determines the distribution of the RV. Therefore, the distribution of a RV can be equivalently found via its mgf.

Example 1

Let X_1 and X_2 be independent RV with uniform distribution on $\{1,2,3,4\}$.

Let $Y = X_1 + X_2$. What's the distributin of Y, i.e., pmf of Y?

Solution:

$$f(x) = \frac{1}{4}, \qquad x = 1,2,3,4 \Rightarrow M_X(t) = E(e^{tX}) = \sum_{x=1}^{4} f(x)e^{tx} = \frac{1}{4}\sum_{x=1}^{4} e^{tx}.$$

$$M_Y(t) = E(e^{tY}) = E\left[e^{t(X_1+X_2)}\right] = E(e^{tX_1})E(e^{tX_2}) \qquad \text{by theorem 5.3-1 on page 191.}$$

$$= \left(\frac{1}{4}\sum_{x=1}^{4} e^{tx_1}\right)\left(\frac{1}{4}\sum_{x=1}^{4} e^{tx_2}\right)$$

$$= \frac{1}{16}e^{2t} + \frac{2}{16}e^{3t} + \frac{3}{16}e^{4t} + \frac{4}{16}e^{5t} + \frac{3}{16}e^{6t} + \frac{2}{16}e^{7t} + \frac{1}{16}e^{8t}.$$

$\Rightarrow$ The pmf of Y, $g(y) = P(Y = y) =$ *the coefficient of* e^{yt}, $y = 2,3,\ldots,8.$

Theorem 5.4-1

If $X_1, X_2, \ldots, X_n$ are independent RVs with respective mgfs $M_{X_i}(t)$ where $-h_i < t < h_i$ for positive number h_i, $i = 1, 2, \ldots, n$. Then the mgf of $Y = \sum_{i=1}^{n} a_i X_i$ is $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$, where $-h_i < t < h_i$, $i = 1, 2, \ldots, n$.

Proof.

$$M_Y(t) = E(e^{tY}) = E\left[\exp(t\sum_{i=1}^{n} a_i X_i)\right] = E(e^{ta_1 X_1} e^{ta_2 X_2} \cdots e^{ta_n X_n})$$

$$= E(e^{a_1 t X_1}) \cdots E(e^{a_n t X_n}) \qquad \leftarrow \text{By theorem 5.3-1, page 191.}$$

$$= \prod_{i=1}^{n} M_{X_i}(a_i t) \qquad \leftarrow M_X(t) = E(e^{tX})$$

A corollary follows immediately, and it will be used in some important examples.

Corollary for Theorem 5.4-1

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a distribution with mgf M(t), where $-h < t < h$, then

(a) The mgf of $Y = \sum_{i=1}^{n} X_i$ *is* $M_Y(t) = \prod_{i=1}^{n} M(t) = [M(t)]^n, \ -h < t < h.$

(b) The mgf of $\overline{X} = \sum_{i=1}^{n} \frac{1}{n} X_i$ is $M_{\overline{X}}(t) = \prod_{i=1}^{n} M(\frac{1}{n}t) = \left[M(\frac{t}{n}) \right]^n, \ -h < \frac{t}{n} < h.$

Example 2 (Page 197)

Let $X_1, X_2, \ldots, X_n$ denote the outcome of n *Bernoulli trials*. Each with probability of success p. The *mgf* of X_i, $i = 1, 2, \ldots, n$ is

$$M(t) = 1 - p + pe^t, \qquad -\infty < t < \infty.$$

Now let $Y = \sum_{i=1}^{n} X_i$, Now compute the *mgf* of Y. Figure out the distribution of Y.

Solution :

$$M_Y(t) = \prod_{i=1}^{n} (1 - p + pe^t) = (1 - p + pe^t)^n.$$

$$\rightarrow Y \sim b(n, p)!!!$$

Theorem 5.4-2

If $X_1, X_2, \ldots, X_n$ are independent chi-square RVs with $r_1, r_2, \ldots, r_n$ degrees of freedom, respectively.

Then $Y = X_1 + X_2 + \cdots + X_n$ is $\chi^2(r_1 + r_2 + \cdots + r_n)$.

Proof.

For each X_i, its mgf is given by

$$M_{X_i}(t) = (1-2t)^{-r_i/2}, \qquad i = 1, \ldots, n. \qquad t < \frac{1}{2}.$$

Hence by the corollary for theorem 5.4-1, we have

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = (1-2t)^{-\frac{1}{2}(r_1 + \cdots + r_n)}$$

$$\Rightarrow Y \sim \chi^2(r_1 + \cdots + r_n)$$

Theorem 3.3-2

If the random variable X is $N(\mu, \sigma^2)$, $\sigma^2 > 0$, then the random variable

$$V = \frac{(X-\mu)^2}{\sigma^2} = Z^2 \text{ is } \chi^2(1).$$

Proof of theorem 3.3-2

Obviously, $Z = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$. Since $V = Z^2$, the cdf for V is given by

$$G(v) = P(Z^2 \le v) = P(-\sqrt{v} \le Z \le \sqrt{v}). \qquad v \ge 0.$$

$$= \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= 2\int_0^{v} \frac{1}{\sqrt{2\pi}} e^{-y/2} d\sqrt{y} \qquad \leftarrow \text{Changing of variable with } z = \sqrt{y}$$

$$= \int_0^{v} \frac{1}{\sqrt{2\pi y}} e^{-y/2} dy.$$

By fundamental theorem of Calculus, its pdf is given by

$$g(v) = G'(v) = \frac{1}{\sqrt{2\pi v}} e^{-v/2} = \frac{1}{\sqrt{2\pi}} v^{1/2-1} e^{-v/2} = \frac{1}{2^{1/2}\Gamma(1/2)} v^{1/2-1} e^{-v/2}, \qquad 0 < v < \infty$$

Hence V follows $\chi^2(1)$.

The next two corollaries combine and extend the results of Theorems 3.3-2 and 5.4-2 and give one interpretation of degrees of freedom.

Corollary

Let $Z_1, \ldots, Z_n$ have standard normal distributions, $N(0,1)$. If $Z_1, \ldots, Z_n$ are independent, then $W = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi^2(n)$.

Proof. Since $Z_i \sim N(0,1)$, $Z_i^2 \sim \chi^2(1)$. $\quad i = 1, 2, \ldots, n$.

By theorem 5.4-2 and 3.3-2, we have $W \sim \chi^2(n)$.

Corollary

If $X_1, \ldots, X_n$ are independent normal RVs with $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \ldots, n$. Then the distribution of

$$W = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2$$

is $\chi^2(n)$.

Proof. Since $X_i \sim N(\mu_i, \sigma_i^2)$, we have $\dfrac{X_i - \mu_i}{\sigma_i} \sim N(0,1)$.

$\Rightarrow Z_i = \dfrac{X_i - \mu_i}{\sigma_i} \sim N(0,1)$. And $Z_1, \ldots, Z_n$ are independent, by the above corollary, we have $W \sim \chi^2(n)$.