AIE6001 Assignment 2: Eigenvalue, Probability, Stochastic Processes

Due date: 11:59 PM, Tuesday, October 21, 2025.

Remark: The Maximum point is 100. For instance, if your grade with bonus point is 95+10=105, your actual grade will be $\min(105, 100) = 100$.

Question 1 (Eigenvalue)**.** *Let $c \in \mathbb{R}$ and A be a $n \times n$ matrix. Suppose the sum of the entries in each column of A is c. Show that c is an eigenvalue of A.* *(10 points)*

Question 2 (Eigenvalue)**.** *Suppose A is a real $n \times n$ skew-symmetric matrix ($A^\top = -A$). Prove that any real eigenvalue of A must be zero.* *(10 points)*

Question 3 (Eigenvalue)**.** *Suppose A is a $n \times n$ matrix with $A^k = 0$ for some positive integer k. Prove that the only eigenvalue of A is 0.* *(10 points)*

Question 4 (SVD)**.** *Show the proof of the low-rank approximation theorem stated in Slides `Lecture3.pdf`. (10 points)*

Question 5 (Events)**.** *Prove that for any given events $A_1, \ldots, A_n$, it holds that*

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{n} P(A_i).$$

Give an example where equality does not hold. *(10 points)*

Question 6 (Probability)**.** *A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:*

1) ***Four of a kind*** *(four cards of equal face value and one card of a different value).*
2) ***Full house*** *(one pair and one triple of cards with equal face value).*
3) ***Three of a kind*** *(three equal face values plus two cards of different values).*
4) ***Two pairs*** *(two pairs of equal face value plus one card of different value).*
5) ***One pair*** *(one pair of equal face value plus three cards of different values).*

(10 points)

Question 7 (pdf)**.** *Let X be a random variabe with pdf $f(x) = \frac{1}{\pi(1+x^2)}, x \in (-\infty, \infty)$. Show that $\mathbb{E}[|X|]$ is infinite, and hence it means that it does not exist.* *(10 points)*

Question 8 (Conditional Probability). *Let X have an exponential distribution with mean $\Theta > 0$. Show that*

$$P(X > x + y \mid X > x) = P(X > y).$$

(10 points)

Question 9 (Ordered Statistics). *Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $Y_n = \min(X_1, \ldots, X_n)$ and $Z_n = \max(X_1, \ldots, X_n)$. Find the probability that both Z_n and Y_n lie in the interval $[a, b]$, where $-\infty < a < b < \infty$.*

(10 points)

Question 10 (Stochastic Process).

1) *Suppose you keep on tossing a fair coin. What is the expected number of tosses such that you can have* ***HHH*** *(heads heads heads) in a row? What is the expected number of tosses to have* ***THH*** *(tails heads heads) in a row?*
2) *Keep flipping a fair coin until either* ***HHH*** *or* ***THH*** *occurs in the sequence. What is the probability that you get an* ***HHH*** *subsequence before* ***THH****?*

(10 points)

Question 11 (Stochastic Process, bonus). *A drunk man is at the 17th meter of a 100-meter-long bridge. He has a 50% probability of staggering forward or backward one meter each step.*

- *What is the probability that he will make it to the end of the bridge (the 100th meter) before the beginning (the 0th meter)?*
- *What is the expected number of steps he takes to reach either the beginning or the end of the bridge?*

(10 points)