1. Newsvendor Problem

Background

- Suppose we sell newspaper to Georgia Tech campus.
- We have to **order** a specific number of copies **from the publisher** every evening and **sell** those copies the next day.
- You want to know what is the **optimal number of copies you need to order** every day

Background

- **Selling price (p):** How much will you charge per paper?
- **Buying price (c):** How much will the publisher charge per paper?
- **Your order quantity (x):** You will decide how many papers to be ordered before you start a day.
- **Customer demand (D):** How many people who want to buy newspaper, following known probability distribution.

$$\text{Formulation:} \quad \max_{x} \; \mathbb{E}[p \cdot \min(x, D) - c \cdot x]$$

How to compute objective?

- For fixed x, how to compute $\mathbb{E}[p \cdot \min(x, D) - c \cdot x]$

Example 1. **Suppose $\boldsymbol{p = 1, c = 0.1, x = 18}$.**
$\boldsymbol{D}$ is a discrete random variable with pmf

d	10	15	20	25	30
$P(D = d)$	1/4	1/8	1/8	1/4	1/4

How to compute objective? (c.n.t.)

- For fixed x, how to compute $\mathbb{E}[p \cdot \min(x, D) - c \cdot x]$

Example 2. Suppose $p = 1, c = 0.1, x = 18$.
D is a continuous random variable following uniform distribution on $[10, 30]$

How to find optimal order quantity?

$$\max_x \; \mathbb{E}[p \cdot \min(x, D) - c \cdot x]$$

Theorem 1. Let D be a continuous demand with cdf $F_D(\cdot)$. Then the optimal order quantity x^* satisfies that

$$F_D(x^*) = \frac{p - c}{p}.$$

Example 3. **Suppose $p = 1, c = 0.1$, and D is a continuous random variable following uniform distribution on $[10, 30]$. What is the optimal ordering quantity?**

How to find optimal order quantity?

$$\max_x \ \mathbb{E}[p \cdot \min(x, D) - c \cdot x]$$

Theorem 2. Let D be a **discrete** demand with cdf $F_D(\cdot)$. Then the optimal order quantity x^* is the smallest number x such that

$$F_D(x) \geq \frac{p-c}{p}.$$

Example 4. Suppose $p = 1, c = 0.1$, and D is a discrete random variable with pmf

d	10	15	20	25	30
$P(D = d)$	3/10	3/10	3/10	5/20	5/20

What is the optimal order quantity?

2. Queuing Theory

Background

- All of you must have experienced waiting in a **service system**. One example would be the Student Center or some restaurants.

Background

- **Inter-arrival Times:** How frequently customers come to your system ?
- **Service Times:** How fast your servers can serve the customers?
- **Number of Servers:** How many servers do you have

In general, a queueing system can be denoted as follows:

G/G/s

Lindley Equation

Let

- u_i be the i-th customer's inter-arrival time
- v_i be the i-th customer’s service time
- w_i be the i-th customer’s waiting time

How to compute the (i+1)-th customer’s waiting time?

Lindley Equation

Let

- u_i be the i-th customer's inter-arrival time
- v_i be the i-th customer’s service time
- w_i be the i-th customer’s waiting time

How to compute the (i+1)-th customer’s waiting time?

- **Lindley equation:** $\boldsymbol{w_{i+1} = (w_i + v_i - u_{i+1})^+}$

Example 2.2. Given the following inter-arrival times and service times of first 10 customers, compute waiting times and system times (time spent in the system including waiting time and service time) for each customer.

$$u_i = 3, 2, 5, 1, 2, 4, 1, 5, 3, 2$$
$$v_i = 4, 3, 2, 5, 2, 2, 1, 4, 2, 3$$

Traffic Intensity

Suppose that

$$\mathbb{E}[u_i] = \text{mean inter-arrival time} = 2 \text{ min}$$

$$\mathbb{E}[v_i] = \text{mean service time} = 4 \text{ min}$$

Is the system stable?

Definition 1. Let the traffic intensity be

$$\rho = \frac{\lambda}{\mu},$$

where $\lambda = \frac{1}{E[u_i]}$ denotes the arrival rate, $\mu = \frac{1}{E[v_i]}$ denotes the service rate.

Traffic Intensity

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1. If $\rho < 1$, the system is **stable**. Then, ρ is the long-run fraction of time where the server is utilized.
2. If $\rho > 1$, the system is **unstable**.

A small bank is staffed by a single server. It has been observed that, during a normal business day, the inter-arrival times of customers to the bank are iid having exponential distribution with mean 3 minutes. Also, the the processing times of customers are iid having the following distribution (in minutes):

x	1	2	3
$\mathbf{Pr}\{X = x\}$	1/4	1/2	1/4

An arrival finding the server busy joins the queue. The waiting space is infinite.

What is the long-run fraction of time that the server is busy?

3. Discrete Time Markov Chain

Introduction

- Suppose your daily mood only depends on your mood on the previous day.
- State space: S ={**Happy (1)**, **So-so (2)**, **Gloomy (3)** }.
- Transition probability matrix:

$$\begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.9 & 0.1 \end{pmatrix}$$

- **Denote by X_n the mood at the n-th day. Here, the process $\{X_n\}_n$ is a discrete-time Markov chain (DTMC).**
- **What is the probability $P(X_1 = 2 \mid X_0 = 1)$?**
- **What is the probability $P(X_2 = 2 \mid X_0 = 1)$?**

Stationary Distribution

Definition 2. Suppose $\pi = (\pi_i, i \in S)$ satisfies

- $\pi_i \geq 0, \forall i \in S$
- $\sum_i \pi_i = 1$
- $\pi = \pi P$

Then, π is said to be a stationary distribution of the DTMC.

What is the stationary distribution for

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.9 & 0.1 \end{pmatrix}?$$

What does π_2 mean?

Stationary Distribution (Example)

What is the stationary distribution for

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.9 & 0.1 \end{pmatrix}?$$

What does π_2 mean?

4. Poisson Process

Homogeneous Poisson Process

- A sequence of customers having i.i.d. exponential inter-arrival time with **constant rate λ** is a Homogeneous Poisson Process.
- Let $N(t)$ be the number of arrivals of customers in [0,t].

  1. $N(t) - N(s) \sim Poisson\big(\lambda \cdot (t - s)\big), \forall t > s$.
  2. $N(t) - N(s)$ is independent of $N(u), \forall s \geq u$.
  3. $N(0) = 0$.

Example

Assume $\{N(t)\}_t$ is a Poisson process with rate $\lambda = 2$.

1. Probability that there are exactly 4 arrivals in first 3 minutes?
2. Probability that there is no arrival in [0,4]?
3. Probability that the first arrival will take at least 4 minutes?