

Lecture 4

Basics of Probability

- Probability
- Conditional Probability
- Discrete and Continuous Distributions
- Conditional Expectation and Variance
- Order Statistics

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Elements of Probability

- Outcome (ω): the outcome of an experiment or trial
- Sample space/Probability space (Ω): the set of all possible outcomes of an experiment
- $P(\omega)$: the probability of an outcome
- $P(\omega) \geq 0, \forall \omega \in \Omega, \sum_{\omega \in \Omega} P(\omega) = 1.$
- Random variable: a function that maps each outcome (ω) in the sample space (Ω) into the set of real numbers.

Event

- Event: A set of outcomes, a subset of the sample space.
- $P(A)$:Probability of an event A , i.e., $P(A) = \sum_{\omega \in A} P(\omega)$.
- $A \cup B$: set of outcomes in event A or in event B (or both)
- $A \cap B$: set of outcomes in both A and B , also written as AB
- A^c : the complement of A , which is the event “not A ”
- Mutually exclusive: $A \cap B = \emptyset$.
- For any mutually exclusive events E_1, \dots, E_N ,

$$P\left(\bigcup_{i=1}^N E_i\right) = \sum_{i=1}^N P(E_i).$$

Example

Consider rolling a six-sided dice.

- $\Omega = \{1, 2, 3, 4, 5, 6\}$, and the probability of each outcome is $1/6$.
- Assume that $A = \{1, 3, 5\}$. Calculate $A^c, P(A)$.
- Let B denote the event that the outcome is larger than 3. Calculate $A \cup B, A \cap B$.
- Define a special random variable I_A such that

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in \{1, 3, 5\}, \\ 0, & \text{if } \omega \notin \{1, 3, 5\}. \end{cases}$$

Calculate $\mathbb{E}[I_A]$.

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Example

Two gamblers are playing a coin toss game. Suppose Gambler A has 2 fair coins and B has 1 fair coin. What is the probability that A will have more heads than B if both flip all their coins?

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Definition

- Conditional probability: If $P(B) > 0$, then $P(A | B) = \frac{P(A \cap B)}{P(B)}$ is the fraction of B outcomes that are also A outcomes.
- Multiplication rule:

$$\begin{aligned} &P(E_1 E_2 \cdots E_n) \\ &= P(E_1) P(E_2 | E_1) P(E_3 | E_1 E_2) \cdots P(E_n | E_1 \cdots E_{n-1}). \end{aligned}$$

- Law of total probability: for any mutually exclusive events $\{F_i\}, i = 1, \dots, n$ whose union is the entire sample space,

$$\begin{aligned} P(E) &= P(EF_1) + \cdots + P(EF_n) \\ &= P(E | F_1)P(F_1) + \cdots + P(E | F_n)P(F_n) \end{aligned}$$

Definition

- Independent events: $P(EF) = P(E)P(F)$. Also,

$$P(EF^c) = P(E)P(F^c).$$

- Bayes' formula:

$$P(F_j | E) = \frac{P(E | F_j)P(F_j)}{\sum_{i=1}^n P(E | F_i)P(F_i)},$$

provided that $F_i, i = 1, \dots, n$ are mutually exclusive events whose union is the entire sample space.

Example

1. A company is holding a dinner for working mothers **with at least one son**. Ms. Jackson, a mother with two children, is invited. What is the probability that both children are boys?
2. Your new colleague, Ms. Parker is known to have two children. Suppose you see her walking with **one of her children and that child is a boy**. What is the probability that both children are boys?
3. Part 1 asks given there is at least one boy in two children, what is the conditional probability that both children are boys. Part 2 asks that given one child is a boy, what is the conditional probability that the other child is also a boy.

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Example

In a primitive society, let's assume every couple prefers to have a baby boy. Assume there is a 50% chance that each child they have is a boy, and the genders of their children are mutually independent. If each couple insists on having more children until they get a boy and once they have a boy, they will stop having more children. What will eventually happen to the fraction of boys in this society?

Example

You are given 1000 coins. Among them, 1 coin has heads on both sides. The other 999 coins are fair coins. You randomly choose a coin and toss it 10 times. Each time, the coin turns up heads. What is the probability that the coin you choose is the unfair one?

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Common functions of random variables

Random Variable X	Discrete	Continuous
Cumulative distribution function (cdf)	$F(a) = P\{X \leq a\}$	$F(a) = \int_{-\infty}^a f(x)dx$
Probability mass function (pmf) or Probability density function (pdf)	$p(x) = P\{X = x\}$	$f(x) = \frac{d}{dx}F(x)$
Expected value $\mathbb{E}[X]$	$\sum_{x: p(x)>0} xp(x)$	$\int_{-\infty}^{\infty} xf(x)dx$
Expected value of $g(x)$, $\mathbb{E}[g(x)]$	$\sum_{x: p(x)>0} g(x)p(x)$	$\int_{-\infty}^{\infty} g(x)f(x)dx$
Variance of X , $\text{Var}(X)$	$\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$	
Standard deviation of X , $\text{std}(X)$	$\sqrt{\text{Var}(X)}$	

Common discrete random variables

Name	Probability mass function (pmf)	$E[X]$	$\text{var}(X)$
Uniform	$P(x) = \frac{1}{b-a+1}, \quad x = a, a+1, \dots, b$	$\frac{b+a}{2}$	$\frac{(b-a+1)^2 - 1}{12}$
Binomial	$P(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$	np	$np(1-p)$
Poisson	$P(x) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}, \quad x = 0, 1, \dots^{20}$	λt	λt
Geometric	$P(x) = (1-p)^{x-1} p, \quad x = 1, 2, \dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial	$P(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \dots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

Common continuous random variables

Name	Probability density function (pdf)	$E[X]$	$\text{var}(X)$
Uniform	$\frac{1}{b-a}, a \leq x \leq b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in (-\infty, \infty)$	μ	σ^2
Exponential	$\lambda e^{-\lambda x}, x \geq 0$	$1/\lambda$	$1/\lambda^2$
Gamma	$\frac{\lambda e^{-\lambda x} (\lambda x)^{a-1}}{\Gamma(a)}, x \geq 0, \Gamma(a) = \int_0^\infty e^{-y} y^{a-1}$	a/λ	a/λ^2
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, 0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

Table 4.2: Common continuous random variables

Example

If X is a standard normal random variable, what is $\mathbb{E}[X \mid X > 0]$?

Example

Two bankers each arrive at the station at some random time between 5:00am and 6:00am (arrival time for either banker is uniformly distributed). They stay exactly five minutes and then leave. What is the probability they will meet on a given day?

Example

A stick is cut twice randomly (each cut point follows a uniform distribution on the stick). What is the probability that the 3 segments can form a triangle?

Example

You are waiting for a bus at a bus station. The bus arrive at the station according to a Poisson process with an average arrival time of 10 minutes ($\lambda = 0.1/\text{min}$). If the buses have been running for a long time and you arrive at the bus station at a random time, what is your expected waiting time? On average, how many minutes ago did the last bus leave?

Example

Assume X follows standard normal distribution, then what is $\mathbb{E}[X^n]$ for $n = 1, 2, 3, 4$?

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Definition

- For discrete distribution,

$$\mathbb{E}[g(X) \mid Y = y] = \sum_x g(x)p_{X|Y}(x \mid y) = \sum_x g(x)p(X = x \mid Y = y).$$

- For continuous distribution,

$$\mathbb{E}[g(X) \mid Y = y] = \int_{-\infty}^{\infty} g(x)f_{X|Y}(x \mid y)dx.$$

- Law of total expectation:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \begin{cases} \sum_y \mathbb{E}[X \mid Y = y]p(Y = y), & \text{for discrete } Y, \\ \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y]f_Y(y)dy, & \text{for continuous } Y. \end{cases}$$

Example

You just brought one share of stock A and wish to hedge it by shorting stock B . How many shares of B should you short in order to minimize the variance of the hedged position? Assume that the variance of stock A 's return is σ_A^2 , and that of B 's return is σ_B^2 . Their correlation coefficient is ρ .

Example

Suppose that you roll a dice. For each roll, you are paid the face value. if the roll gives 4, 5, or 6, you can roll the dice again. Once you get 1, 2, or 3, the game stops. What is the expected payoff of this game?

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Definition

- Let X be a random variable with cumulative distribution function $F_X(x)$.
- Define $Y_n = \min(X_1, \dots, X_n)$ and $Z_n = \max(X_1, \dots, X_n)$.
- $f_{Y_n}(x) = nf_X(x)(1 - F_X(x))^{n-1}$
- $f_{Z_n}(x) = nf_X(x)(F_X(x))^{n-1}$
- Proof:

Example

Let X_1, \dots, X_n be i.i.d. random variables with uniform distribution between 0 and 1.

- What are the cumulative distribution function, the probability density function, and expected value of $Z_n = \max(X_1, \dots, X_n)$?
- What are the cumulative distribution function, the probability density function, and expected value of $Y_n = \min(X_1, \dots, X_n)$?